A Distributed Algorithm for Average Consensus on Strongly Connected Weighted Digraphs ⋆

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Abstract

In this work we propose a distributed algorithm to solve the discrete-time average consensus problem on strongly connected weighted digraphs (SCWDs). The key idea is to couple the computation of the average with the estimation of the left eigenvector associated to the zero eigenvalue of the Laplacian matrix according to the protocol described in Qu et al. (2012). The major contribution is the removal of the requirement of the knowledge of the out-neighborhood of an agent, thus paving the way for a simple implementation based on a pure broadcast-based communication scheme.

1 Introduction

In the past decades, multi-agent systems have gained an increasing interest from the control theory community. Applications range from transportation to environmental monitoring (see Oh et al. (2007)). Distributed algorithms to estimate the status of the system are essential in this context, as they can help the agents modify their behavior in order to improve the global response, Ren and Beard (2007); Gasparri et al. (2012).

Within several of the works related to this topic, the communication among agents is modeled using an undirected communication graph (see Mesbahi and Egerstedt (2010) and the references therein). This is founded on the assumptions that the communication is isotropic, i.e., the employed antenna radiates its power uniformly in all directions and that its range is the same for all the agents in the network. Therefore, if an agent can communicate with another one, the opposite is possible as well. However, this assumption is not always realistic in a real world scenario due, for example, to environmental effects or the radiation pattern of the agents, Luthy et al. (2007).

In this work, we consider a more general scenario where the communication among the agents is modeled as a directed graph. In particular, two different communication schemes can be considered, that is point-to-point or broadcast. We refer to point-to-point as a communication mechanism where an agent (sender) transmits a specific message to another agent (receiver), picking out exactly that agent among all of his neighbors. Note that this communication scheme requires the sender to know the neighbors it is going to send the messages to, i.e., each agent must know its out- neighborhood. Differently, we refer to broadcast as a communication mechanism were an agent (sender) can simply transmit a message which will be received by any other agent (receivers) within its range of transmission. In our opinion, this latter communication mechanism represents a better choice since it can be more easily implemented and provides a higher robustness to the system.

Our contribution is a novel distributed algorithm to compute the average consensus over any strongly connected weighted digraph, which can be run concurrently with the estimation procedure described in Qu et al. (2012) for the computation of the left eigenvector associated to the zero eigenvalue of the Laplacian matrix and for which agents are not required to be aware of their out-neighborhood. To the best of our knowledge, this work introduces the first approach suitable for an implementation based on a pure broadcast communication scheme.

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2 Related Works

In this section we review the major contributions available at the state of the art concerning the average consensus problem on digraphs.

In Dominguez-Garcia and Hadjicostis (2011), a doubly stochastic weight matrix is computed by an iterative procedure that adjusts the outgoing weights of each node. Notably, the fact that the columns of the weight matrix sums to one at each step, guarantees that the average consensus can be performed in parallel with respect to the convergence of the weight matrix to a doubly stochastic form.

In Cai and Ishii (2012), the average consensus over a directed network topology is addressed. The proposed algorithms require an augmentation of the variables of each agent adding a "surplus" variable to be sent to the different out-neighbors, thus requiring the knowledge of the out-neighborhood.

In Atrianfar and Haeri (2012), the average consensus problem is addressed both in the continuous time and in the discrete time under the assumption of switching network topology. However, the discrete time consensus algorithm requires the adjacency matrix to be doubly stochastic.

In Hadjicostis and Charalambous (2013), the discrete-time average consensus problem in the presence of bounded delays in the communication links and changing interconnections is addressed. The proposed ratio-consensus protocol requires that each agent is aware of the number of its out-neighborhood.

In Dominguez-Garcia and Hadjicostis (2013), the authors present a class of distributed iterative algorithms to asymptotically scale a primitive column stochastic matrix to a double stochastic and demonstrate the application of these algorithms to the average consensus problem. In particular, each node is in charge of assigning weights on its outgoing edges based on the weights on its incoming edges. Thus, the knowledge of the out-neighborhood is required.

Kempe et al. (2003) propose a gossip-based *push-sum* protocol to compute the average based on the assignment of the weights of the out-going neighbors such that their sum is unitary or, in other terms, the knowledge of each agent's out degree is required.

Olshevsky and Tsitsiklis (2009) present two different strategies to compute the average when the graph is not balanced. The first one requires the exact knowledge of the left eigenvector whereas the second one assumes bidirectional communications, i.e., an undirected graph. Compared to these algorithms our approach can be run on any strongly connected digraph without any prior knowledge of its left eigenvector.

Consensus in time-varying digraphs is analyzed in Hendrickx and Tsitsiklis (2013) and Touri (2012), giving conditions on the sequence of graphs to ensure convergence to a weighted average of the initial conditions. However, in order to reach the exact average, the sequence of matrices needs to be doubly stochastic or balanced.

Eventually, in Chen et al. (2010) an approach to solve the average consensus on networks with random packet losses is presented. Differently from our approach, this work requires the agents to send an additional variable keeping track of the changes in the state variables caused by the neighbors influence. However, the assumption on the links failure probabilities implies the existence of bidirectional communications.

3 Preliminaries

Let us consider a set of n agents whose communication network is described by a *digraph* $\mathcal{G}(\mathcal{V},\mathcal{E})$ where $\mathcal{V}=\{1,\cdots,n\}$ is the set of nodes and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is the set of directed edges, i.e., ordered pairs of nodes. Let us define the *weighted adjacency matrix* $\mathcal{A}(\mathcal{G})\in\mathbb{R}^{n\times n}$ as follows: $\mathcal{A}_{ij}(\mathcal{G})>0$ if $(j,i)\in\mathcal{E}$, $\mathcal{A}_{ij}(\mathcal{G})=0$ otherwise. Note that $\mathcal{A}_{ij}(\mathcal{G})>0$ if the agent i can receive data from the agent j. It is worthy to point out that the previously defined adjacency matrix is based on the incoming edges of each node. It is assumed that no self loops exist in the network, i.e., $(i,i)\notin\mathcal{E}$. The in-degree and the out-degree of a node k are given by $d_{in}(k)=\sum_j\mathcal{A}_{kj}(\mathcal{G})$ and $d_{out}(k)=\sum_j\mathcal{A}_{jk}(\mathcal{G})$, respectively. The *Laplacian matrix* is defined as $\mathcal{L}(\mathcal{G})=\mathcal{D}(\mathcal{G})-\mathcal{A}(\mathcal{G})$, with $\mathcal{D}(\mathcal{G})$ the *diagonal in-degree matrix* defined as $\mathcal{D}(\mathcal{G})=\left[d_{in}(1),\ldots,d_{in}(n)\right]^T$. For the sake of readability, the dependency on the graph $\mathcal{G}$ will be omitted in the rest of the paper. Let us recall that the Laplacian matrix is a non-symmetric weakly diagonal dominant matrix. It has a zero structural eigenvalue for which the corresponding right eigenvector is the vector of ones of appropriate size, i.e., $\mathcal{L}\mathbf{1}=\mathbf{0}$.

Let the following assumptions be satisfied throughout the rest of the paper:

- **A1** A unique identifier is associated to each agent i of the network, e.g., the MAC address.
- **A2** Each agent sends n variables.
- **A3** Each agent does not know the number of agents receiving its information (i.e., its out degree).
- **A4** The network topology of the considered multi-agent system is described by a static SCWD.

In **A1**, we assume that each agent can distinguish the information coming from the other agents according to the identifier of the sender. In **A2**, it is assumed that each agent has enough storage size for the values coming from its in-neighbors. Therefore, the number of agents belonging to the network is known by each agent. In **A3**, it is stated that each agent can not count the number of its out-neighbors. Eventually, in **A4** we assume that the information produced by one node is propagated within the network.

4 Decentralized Estimation of the Left Eigenvector

In this section, the distributed procedure for the estimation of left eigenvector associated to the zero structural eigenvalue of the Laplacian matrix encoding a SCWD proposed in Qu et al. (2012) is briefly reviewed.

Let us consider the Perron matrix $\mathcal{C}$ defined as: $\mathcal{C} = \mathcal{I} - \beta\,\mathcal{L}$ with $0 < \beta < \dfrac{1}{\Psi}$ and $\Psi = \max_i\{\sum_{j\neq i}\mathcal{A}_{ij}\}$ and let agent i have a variable $\delta_i(k) = [\delta_{i1}(k)\ \ldots\ \delta_{in}(k)]^T$ with initial values $\delta_{ij}(0) = 1$ if $i = j$, 0 otherwise. At each iteration, the agents update their variables as follows:

$$\delta_{ij}(k+1) = \sum_{p\in\mathcal{N}_i\cup i} \mathcal{C}_{ip}\delta_{pj}(k), \tag{1}$$

with $\mathcal{N}_i = \{j \in \mathcal{V} : (j,i) \in \mathcal{E}\}$ the in-neighborhood of agent i. Note that, update rule (1) can be put in vectorial form as: $\Delta(k+1) = \mathcal{C}\,\Delta(k)$, with $\Delta(k) = [\delta_1(k),\,\ldots,\,\delta_n(k)]^T$. Noting that $\Delta(0) = \mathcal{I}$, it is easy to see that at iteration k, the variable $\delta_i(k)$ contains exactly the value of the i^{th} row of the matrix $\mathcal{C}^k$.

Let us denote with $\lambda_{\mathcal{C}_i}$ and $\lambda_{\mathcal{L}_i}$, the i^{th} eigenvalue of the Perron Matrix $\mathcal{C}$ and of the Laplacian matrix $\mathcal{L}$, respectively, for which holds: $\lambda_{\mathcal{C}_i} = 1 - \beta\,\lambda_{\mathcal{L}_i}$. It follows that the two matrices also share the same set of eigenvectors. In particular for the eigenvalue of maximum modulus of the Perron Matrix $\mathcal{C}$, namely $\lambda_{\mathcal{C}_1} = 1$, to which it corresponds the zero eigenvalue of the Laplacian Matrix $\mathcal{L}$, namely $\lambda_{\mathcal{L}_1} = 0$ we have that $\mathcal{C}\,\mathbf{1} = \lambda_{\mathcal{C}_1}\mathbf{1}$ and $w^T\,\mathcal{C} = \lambda_{\mathcal{C}_1} w^T$, with w^T the left eigenvector associated to $\lambda_{\mathcal{C}_1}$ and $\lambda_{\mathcal{L}_1}$.

From the Perron-Frobenius theorem it follows that if the graph is strongly connected by applying the update rule given in (1), then $\lim_{k\to\infty}\Delta(k) = \dfrac{\mathbf{1}\,w^T}{w^T\mathbf{1}}$ or, in other terms, $\delta_i(k)$ will tend to the normalized left eigenvector w of the Laplacian matrix encoding the digraph.

5 Average Consensus Algorithm

In this section we present a solution to the average consensus problem on a general SCWD by using the left eigenvector estimation algorithm described in Section 4. For the sake of simplicity and without loss of generality, let us assume each agent $i \in \mathcal{V}$ has a scalar state, namely $x_i(k) \in \mathbb{R}$, and the elements of the adjacency matrix to be unitary. Let us refer to $x(k) \in \mathbb{R}^n$ as the state vector $x(k) = [x_1(k)\ x_2(k)\ \ldots\ x_n(k)]^T$ and to $x(0) \in \mathbb{R}^n$ as the initial conditions of the system $x(0) = [x_1(0)\ x_2(0)\ \ldots\ x_n(0)]^T$.

Briefly speaking, the average consensus on a digraph is the problem of computing $\mu = \sum_i x_i(0)/n$, where each agent uses only its locally available information. The discrete time update law used to solve the consensus problem is given by the following equation:

$$x_i(k+1) = x_i(k) + \beta \sum_{j\in\mathcal{N}_i}\big(x_j(k) - x_i(k)\big). \tag{2}$$

Let us remark that, for each agent, the required information to compute (2) is obtained by its in-neighbors. The previous equation can be rearranged in terms of the product between a matrix and a vector as $x(k+1) = \mathcal{C}\,x(k)$.

It is a well established result that with a balanced digraph, the classical consensus algorithm leads to an average consensus, Carli et al. (2008). Unfortunately, the same statement does not hold for the general case of digraphs, where the consensus value is given by $\bar{\mu} = \sum_i w_i x_i(0) \neq \mu$, being w_i the i^{th} coefficient of the left eigenvector $w = \mathbf{1}$ associated to the eigenvalue $\lambda_{\mathcal{C}_1}$ introduced before.

In order to reach the average in the case of a general SCWD, the actual initial conditions $x(0)$ can be opportunely modified as $\tilde{x}(0) = x(0) + \Gamma$ so that:

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i(0) = \sum_{i=1}^{n} w_i\,\big(x_i(0) + \Gamma_i\big), \tag{3}$$

with $\Gamma = [\Gamma_1\ \ldots\ \Gamma_n]^T$ the extra term that needs to be adjusted. In particular, each component of the initial conditions is required to satisfy $\dfrac{x_i(0)}{n} = w_i\,\big(x_i(0) + \Gamma_i\big)$, which leads to

$$\Gamma_i = x_i(0)\left(\frac{1}{n\,w_i} - 1\right) = x_i(0)\left(\frac{1 - n\,w_i}{n\,w_i}\right). \tag{4}$$

Let us now assume the left eigenvector to be available at time $k = 0$. This implies that the vector Γ can also be computed. Therefore, the average consensus over a digraph can be achieved by following two different approaches:

(1) fixing the initial conditions $x(0)$ before starting the algorithm,
(2) injecting a suitable exogenous input at any given step k.

In the sequel, an algorithm based on the second strategy is described. To this end, let us first introduce the following proposition.

Proposition 1 *The correction term Γ can be equivalently injected at any iteration k, that is*

$$\langle x(0) + \Gamma,\, w\rangle = \langle x(k) + \Gamma,\, w\rangle\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$.

Proof: To prove the proposition let us recall the well-known property of the left eigenvector for discrete time systems $\langle x(k),\, w\rangle = \lambda_{\mathcal{C}_1}^k\,\langle x(0),\, w\rangle$. Thus, by linearity of the inner product and being $\lambda_{\mathcal{C}_1} = 1$, it follows that:

$$\begin{aligned}\langle x(0) + \Gamma,\, w\rangle &= \langle x(0),\, w\rangle + \langle \Gamma,\, w\rangle\\ &= \langle x(k),\, w\rangle + \langle \Gamma,\, w\rangle\\ &= \langle x(k) + \Gamma,\, w\rangle\,.\end{aligned}$$

■

By assuming the estimate of the eigenvector w to be asymptotic, it follows that a possible technique to asymptotically achieve the consensus is to modify the update rule given in eq. (2) as follows:

$$x_i(k+1) = x_i(k) + \epsilon_i(k) + \beta \sum_{j \in \mathcal{N}_i} \Big(x_j(k) + \epsilon_j(k) - x_i(k) - \epsilon_i(k) \Big) \tag{5}$$

where the iterative error $\epsilon_i(k)$ is defined as:

$$\epsilon_i(k) = \tilde{\Gamma}_i(k) - \tilde{\Gamma}_i(k-1) \tag{6}$$

with:

$$\tilde{\Gamma}_i(k) = x_i(0) \left(\frac{1}{n\, \delta_{ii}(k)} - 1 \right) \tag{7}$$

and $\tilde{\Gamma}_i(-1) = 0$. Clearly, from an implementation standpoint, each agent i broadcasts the quantity $\hat{x}_i(k) = x_i(k) + \epsilon_i(k)$ at each time step k.

The modified consensus algorithm can be expressed in vector form as:

$$x(k+1) = \mathcal{C} \Big(x(k) + \epsilon(k) \Big) \tag{8}$$

with $\epsilon(k) = [\epsilon_1(k) \, \dots \, \epsilon_n(k)]^T$ the error vector at time k. Algorithm 1 shows the pseudo-code of the k^{th} iteration of the average consensus algorithm run by the i^{th} agent.

Algorithm 1 Average Consensus Algorithm

Require: $\beta,\ x_i(k),\ \delta_i(k),\ \{x_j(k)\},\ \{\delta_j(k)\}\ j \in \mathcal{N}_i$
Ensure: $x_i(k+1)$

`/* Update left eigenvector estimate */`

$\delta_i(k) \leftarrow \delta_i(k-1) + \sum_{j \in \mathcal{N}} \beta(\delta_j(k-1) - \delta_i(k-1))$

`/* Compute exogenous input */`

$\epsilon_i(k) \leftarrow \frac{x_i(0)}{n} \left(\frac{\delta_{ii}(k-1) - \delta_{ii}(k)}{\delta_{ii}(k-1)\delta_{ii}(k)} \right)$

`/* Update consensus estimate */`

$x_i(k+1) \leftarrow x_i(k) + \epsilon_i(k) +$
$+ \beta \sum_{j \in \mathcal{N}_i} \Big(x_j(k) + \epsilon_j(k) - x_i(k) - \epsilon_i(k) \Big)$

Proposition 2 *Let us assume the multi-agent system applies the modified consensus algorithm given in eq.* (8). *Then it follows that:*

$$\lim_{k \to \infty} \langle x(k),\, w \rangle = \langle x(0),\, w \rangle + \langle \Gamma,\, w \rangle \tag{9}$$

Proof: Let us consider the update at time k

$$\begin{aligned} \big\langle x(k+1),\, w \big\rangle &= \big\langle x(k) + \epsilon(k),\, w \big\rangle \\ &= \big\langle x(k),\, w \big\rangle + \big\langle \epsilon(k),\, w \big\rangle \\ &= \big\langle x(k-1) + \epsilon(k-1),\, w \big\rangle + \big\langle \epsilon(k),\, w \big\rangle \\ &= \big\langle x(0),\, w \big\rangle + \big\langle \sum_{i=0}^{k} \epsilon(i),\, w \big\rangle \end{aligned}$$

At this point, by noticing that the term $\sum_{i=0}^{k} \epsilon(i) = \tilde{\Gamma}(k)$ is a telescoping series it follows that:

$$\langle x(k+1),\, w \rangle = \langle x(0),\, w \rangle + \big\langle \tilde{\Gamma}(k),\, w \big\rangle$$

Furthermore by adding and subtracting the quantity $\langle \Gamma,\, w \rangle$ to the right-hand side, it follows:

$$\langle x(k+1),\, w \rangle = \langle x(0),\, w \rangle + \langle \Gamma,\, w \rangle + \langle \zeta(k),\, w \rangle$$

where $\zeta(k) = [\zeta_1(k),\, \dots,\, \zeta_n(k)]^T$ with $\zeta_i(k) = \tilde{\Gamma}_i(k) - \Gamma_i$. Successively, due to the convergence properties of the left eigenvector estimation algorithm we have:

$$\lim_{k \to \infty} \zeta_i(k) = 0, \quad \forall\, i \in \mathcal{V}, \tag{10}$$

from which it follows:

$$\lim_{k \to \infty} \langle x(k),\, w \rangle = \langle x(0),\, w \rangle + \langle \Gamma,\, w \rangle\,,$$

thus proving the statement. ■

The following proposition provides a bound on the disagreement vector $\varphi(k)$ defined as $\varphi(k) = \mathbf{x}(k) - \mu \mathbf{1}$.

Proposition 3 *Let us assume the multi-agent system applies the modified consensus algorithm give in eq.* (8). *Then, the disagreement vector $\varphi(k)$ can be bounded as:*

$$\|\varphi(k)\| \leq \chi_1\, k |\lambda_{\mathcal{C}_2}|^k + \chi_2 |\lambda_{\mathcal{C}_2}|^k, \tag{11}$$

with $\|\cdot\|$ the Euclidean norm and $\chi_1,\, \chi_2 \in \mathbb{R}$ two positive constant values.

Proof: From eq. (8) at the k^{th} iteration we have $x(k) = \mathcal{C}^k x(0) + \sum_{j=0}^{k-1} \mathcal{C}^{k-j} \epsilon(j)$. From eqs. (3), (6) and (7) the average can be put in the form

$$\mu = \mathbf{w}^T x(0) + \sum_{j=0}^{\infty} \mathbf{w}^T \epsilon(j).$$

The norm of the disagreement vector can be written as:

$$\|\varphi(k)\| \leq \underbrace{\Big\| \mathcal{C}^k x(0) - \mathbf{w}^T x(0)\, \mathbf{1} \Big\|}_{t_1} + \underbrace{\Big\| \sum_{j=0}^{k-1} \mathcal{C}^{k-j} \epsilon(j) - \sum_{j=0}^{\infty} \mathbf{w}^T \epsilon(j)\, \mathbf{1} \Big\|}_{t_2}.$$

As in Montijano et al. (2013), let $\mathcal{Q} = \mathcal{C} - \mathbf{1}\mathbf{w}^T$, whose eigenvalues are $\lambda_{\mathcal{Q}_1} = 0$, with $\mathbf{w}$ and $\mathbf{w}^T x(0) \mathbf{1}$ its corresponding left and right eigenvectors respectively, while the rest of eigenvalues and eigenvectors are the same as for $\mathcal{C}$. At this point, since $\mathcal{C}^k \big(\mathbf{w}^T x(0)\big) \mathbf{1} = \big(\mathbf{w}^T x(0)\big) \mathbf{1}$

and $\mathbf{1}\mathbf{w}^T(x(0) - \mathbf{w}^T x(0)\mathbf{1}) = \mathbf{0}$, we can see that $\mathcal{C}^k(x(0) - \mathbf{w}^T x(0)\mathbf{1}) = \mathcal{Q}^k(x(0) - \mathbf{w}^T x(0)\mathbf{1})$, for all $k \in \mathbb{N}$, and therefore:

$$\begin{aligned} t_1 &= \|\mathcal{Q}^k x(0) - \mathbf{w}^T x(0)\mathbf{1}\| \le \|\mathcal{Q}\|^k \|x(0) - \mathbf{w}^T x(0)\mathbf{1}\| \\ &\le \gamma_1 |\lambda_{\mathcal{C}_2}|^k \|x(0) - \mathbf{w}^T x(0)\mathbf{1}\| \le \chi_{21} |\lambda_{\mathcal{C}_2}|^k, \end{aligned} \tag{12}$$

where $\chi_{21} = \gamma_1 \|x(0) - \mathbf{w}^T x(0)\mathbf{1}\|$ is a constant value with γ_1 another constant due to the diagonalization of $\mathcal{C}$. The following holds for the second term t_2:

$$\begin{aligned} t_2 &= \left\| \sum_{j=0}^{k-1} \left(\mathcal{C}^{k-j}\epsilon(j) - \mathbf{w}^T \epsilon(j)\,\mathbf{1}\right) - \sum_{j=k}^{\infty} \mathbf{w}^T \epsilon(j)\,\mathbf{1} \right\| \\ &\le \underbrace{\left\| \sum_{j=0}^{k-1} \left(\mathcal{C}^{k-j}\epsilon(j) - \mathbf{w}^T \epsilon(j)\,\mathbf{1}\right) \right\|}_{t_{21}} + \underbrace{\left\| \sum_{j=k}^{\infty} \mathbf{w}^T \epsilon(j)\,\mathbf{1} \right\|}_{t_{22}}. \end{aligned}$$

Regarding the term t_{21} we have that:

$$\begin{aligned} t_{21} &\le \sum_{j=0}^{k-1} \left\| \mathcal{C}^{k-j}\epsilon(j) - \mathbf{w}^T \epsilon(j)\,\mathbf{1} \right\| \\ &\le \sum_{j=0}^{k-1} \left\| \mathcal{C} \right\|^{k-j} \left\| \epsilon(j) - \mathbf{w}^T \epsilon(j)\,\mathbf{1} \right\| \\ &\le \sum_{j=0}^{k-1} \gamma_1 |\lambda_{\mathcal{C}_2}|^{k-j} \underbrace{\left\| \epsilon(j) - \mathbf{w}^T \epsilon(j)\,\mathbf{1} \right\|}_{t_{211}}, \end{aligned}$$

where:

$$\begin{aligned} t_{211} &\le \sqrt{n} \left\| \epsilon(j) - \mathbf{w}^T \epsilon(j)\,\mathbf{1} \right\|_\infty \\ &\le \sqrt{n} \max_i \left| x_i(0) - w_i x_i(0) \right| \max_i \left| \frac{\delta_{ii}(j-1) - \delta_{ii}(j)}{n\,\delta_{ii}(j)\,\delta_{ii}(j)} \right| \\ &\le \sqrt{n} \left\| x(0) - \mathbf{w}^T x(0)\mathbf{1} \right\| \underbrace{\max_i \left| \frac{\delta_{ii}(j-1) - \delta_{ii}(j)}{n\,\delta_{ii}(j)\,\delta_{ii}(j)} \right|}_{t_{2111}}. \end{aligned}$$

Since $\delta_{ii}(j)\mathbf{1} = C^j e_i$ with e_i the i-th vector of the canonical basis, thus the following holds for the term t_{2111}:

$$\begin{aligned} t_{2111} &= \max_i \frac{\left|\delta_{ii}(j-1) - w_i + w_i - \delta_{ii}(j)\right|}{\left|n\,\delta_{ii}(j-1)\,\delta_{ii}(j)\right|} \\ &\le \max_i \frac{\left\|C^j e_i - w_i\,\mathbf{1}\right\| + \left\|C^{j-1} e_i - w_i\,\mathbf{1}\right\|}{n\,\bar{d}^2} \\ &\le \max_i \frac{\gamma_1 \left(|\lambda_{\mathcal{C}_2}|^j + |\lambda_{\mathcal{C}_2}|^{j-1}\right) \left\|e_i - w_i\,\mathbf{1}\right\|}{n\,\bar{d}^2} \\ &\le \frac{\gamma_1 |\lambda_{\mathcal{C}_2}|^j \left(1 + \dfrac{1}{|\lambda_{\mathcal{C}_2}|}\right)\sqrt{2}}{n\,\bar{d}^2}, \end{aligned}$$

where the fact $\left\|e_i - w_i\,\mathbf{1}\right\| \le \sqrt{2}$ has been used and $\bar{d} = \min_{i\in\mathcal{V}} \min_{j\in\mathbb{N}} \delta_{ii}(j) > 0$.

Therefore, the term t_{211} is bounded by:

$$t_{211} \le \chi_{11} |\lambda_{\mathcal{C}_2}|^j, \tag{13}$$

with a constant χ_{11} defined as:

$$\chi_{11} = \frac{\gamma_1 \left(1 + \dfrac{1}{|\lambda_{\mathcal{C}_2}|}\right)\sqrt{2}}{n\,\bar{d}^2} \left\| x(0) - \mathbf{w}^T x(0)\mathbf{1} \right\|, \tag{14}$$

Therefore:

$$t_{21} \le \sum_{j=0}^{k-1} \gamma_1 \chi_{11} |\lambda_{\mathcal{C}_2}|^k \le \gamma_1 \chi_{11} k |\lambda_{\mathcal{C}_2}|^k, \tag{15}$$

and noting that $\chi_1 = \gamma_1 \chi_{11}$ is also constant the first term of the right hand side of (11) is found.

Regarding the term t_{22}, following the same reasoning as in eq. (13), we have that:

$$\begin{aligned} \left\| t_{22} \right\| &\le \left\| \sum_{j=k}^{\infty} \mathbf{w}^T \epsilon(j)\,\mathbf{1} \right\| \\ &\le \sqrt{n} \max_i \left| \lim_{j\to\infty} w_i \tilde{\Gamma}_i(j) - w_i \tilde{\Gamma}_i(k-1) \right| \\ &\le \max_i \frac{\sqrt{n} x_i(0)}{n} \left| \frac{\delta_{ii}(k-1) - w_i}{\delta_{ii}(k-1)} \right| \\ &\le \max_i \frac{\sqrt{n} x_i(0)}{\bar{d}\,n} \left\| C^{k-1} e_i - w_i \mathbf{1} \right\| \\ &\le \frac{\gamma_1 \|x(0)\| \sqrt{2}}{\bar{d}\sqrt{n} |\lambda_{\mathcal{C}_2}|} |\lambda_{\mathcal{C}_2}|^k \le \chi_{22} |\lambda_{\mathcal{C}_2}|^k, \end{aligned} \tag{16}$$

with χ_{22} another constant. At this point, by collection all the terms given in eqs (12), (15) and (16) the bound given in eq. (11) is obtained with $\chi_2 = \chi_{21} + \chi_{22}$. ■

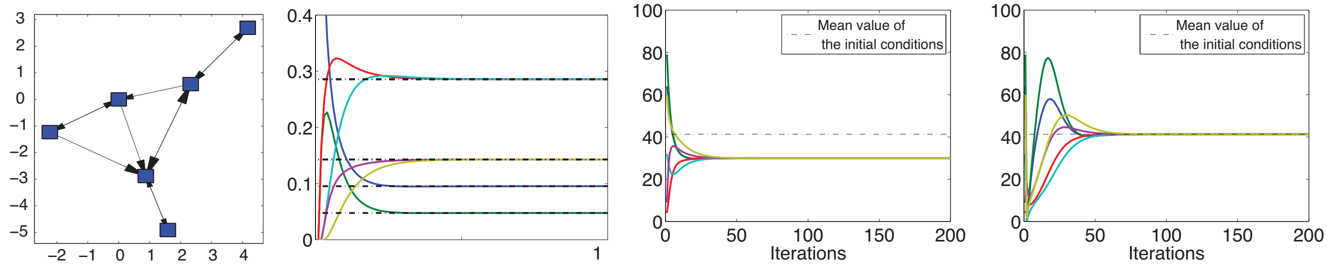
It is now possible characterize the convergence rate of the proposed algorithm as follows.

Corollary 4 *The convergence rate of the algorithm is:*

$$r_{asym} = \sup_{x(0)\ne\mu} \lim_{k\to\infty} \left(\frac{\|\varphi(k)\|}{\|\varphi(0)\|} \right)^{1/k} = |\lambda_{\mathcal{C}_2}| \tag{17}$$

Proof: The proof follows directly from the application of the result given in Proposition 3.

Notably, in our framework the modified consensus algorithm runs concurrently with the left eigenvector estimation protocol and since the two algorithms exhibit the same convergence rate, it follows that no real overhead is introduced in terms of number of exchanged packets at the cost of a slightly larger payload.



(a) Communication Graph (b) Left Eigenvector Estimation (c) Standard Consensus Algorithm (d) Modified Consensus Algorithm

Fig. 1. Simulation involving a multi-agent system composed of 6 agents. Fig. 1-(a) depicts the underlying communication graph describing the interaction among the six agents. Fig. 1-(b) shows the left eigenvector estimation carried out by agent 1. Fig. 1-(c) shows the standard consensus protocol, where the dotted line represents the average value. Fig. 1-(d) shows the proposed modified consensus protocol.

6 Simulations

In this section, a simulation involving a network of 6 agents is considered. The agents run both the classical discrete time consensus and the modified one proposed in this work along with the left eigenvector estimation. Fig. 1(a) depicts the agents network topology. The initial conditions of the agents are $x(0) = [64\ 79\ 4\ 32\ 9\ 60]^T$, leading to $\mu = 41.3333$. Fig. 1(b) depicts the estimation process of the left eigenvector associated to the zero eigenvalue of the Laplacian matrix for Agent 1. Fig. 1(c) depicts the execution of the standard consensus algorithm. It is worthy to notice that in this case the agents can achieve a consensus because the graph is a SCD, but the consensus value is different from μ. Instead, in Fig. 1(d) the execution of the consensus using the modified algorithm is given. In this case, all the agents obtain μ as their consensus value.

7 Conclusion

In this work we proposed a distributed algorithm for the average consensus on any strongly connected weighted digraph without the knowledge of the agents out-neighborhood. The key idea is to couple the computation of the average with the estimation of the left eigenvector associated to the zero eigenvalue of the Laplacian matrix according to the estimation procedure described in Qu et al. (2012). Future work will be focused on the extension of the proposed approach to work on general time-varying digraphs.

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