Average Consensus on Strongly Connected Weighted Digraphs: A Generalized Error Bound

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Abstract

This technical communique represents a generalization of the convergence analysis for the consensus algorithm proposed in Priolo et al. (2014). Although the consensus was reached for any strongly connected weighted digraphs (SCWD), the convergence analysis provided in Priolo et al. (2014) was only valid for diagonalizable matrices encoding a SCWD. The result we present here generalizes the previous one to all possible matrices encoding a SCWD that can be used in the algorithm.

1 Introduction

The problem of distributed average consensus over strongly connected weighted digraphs has received a lot of attention over the past few years. While this problem is well known to be solved for undirected graphs (see Mesbahi and Egerstedt (2010) and references therein), solutions are largely unknown for the remaining cases when graphs are not balanced. This implies that, even when a standard linear iteration will reach consensus, the final value will be some weighted combination of the initial conditions, different to the average. Although reaching a consensus to a certain value might suffice in several application scenarios, such as in context of multi-robot systems, e.g., consensus-based formation control or rendez-vous, reaching the average of the initial conditions is mandatory in some specific scenarios, such as in the context of maximum likelihood estimation in Xiao et al. (2005), or clock synchronization, e.g., Carli et al. (2011); He et al. (2013, 2014). Thus, the design of a distributed algorithm for reaching the average consensus over SCWD is certainly of interest.

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Some approaches dealing with this problem present distributed algorithms that generate a weight-balanced matrix, Dominguez-Garcia and Hadjicostis (2011); Gharesifard and Cortés (2012). Once this matrix is available, a standard linear iteration reaches the average of the initial conditions in the same way as for undirected graphs.

Other methods are based on the introduction of correction terms, e.g., Priolo et al. (2014); Cai and Ishii (2012), that compensate for the errors that the linear iteration introduces in the computation of the consensus. The main contribution of the algorithm in Priolo et al. (2014) was lifting the requirement of the out-neighborhood knowledge for the different agents, making the approach suitable for an implementation based on a pure broadcast communication scheme.

In Priolo et al. (2014), the convergence of the algorithm was proved by following the approach used in Montijano et al. (2013) for which the weight matrix must be diagonalizable. In this technical communique we extend the convergence analysis to the general case of any row stochastic matrix encoding a SCWD.

2 Algorithm Overview

Consider a set of n agents with some initial values $x(0) = [x_1(0)\ x_2(0)\ \dots\ x_n(0)]^T \in \mathbb{R}^n$, with average equal to μ, and interactions between them defined according to a SCWD, encoded by the matrix $\mathcal{C}$. This matrix is defined to be row stochastic (all its rows sum 1), which implies that has one eigenvalue $\lambda_1 = 1$ with multiplicity equal to one, and right and left eigenvectors equal to $\mathbf{1}$ and $\mathbf{w}$ respectively, i.e., $\mathcal{C}\mathbf{1} = \mathbf{1}$ and

$\mathbf{w}^T\mathcal{C} = \mathbf{w}^T$. The modulus of the remaining eigenvalues of $\mathcal{C}$ is strictly less than one, $|\lambda_i| < 1$, for all $i \neq 1$.

The distributed algorithm to reach average consensus over a SCWD is

$$x(k+1) = \mathcal{C}\left(x(k) + \epsilon(k)\right), \tag{1}$$

with $x(k) = [x_1(k)\ x_2(k)\ \ldots\ x_n(k)]^T$ the current estimations of the agents and $\epsilon(k) \in \mathbb{R}^n$ the iterative correction term to reach the average. The individual components of this last vector are

$$\epsilon_i(k) = \tilde{\Gamma}_i(k) - \tilde{\Gamma}_i(k-1), \tag{2}$$

with

$$\tilde{\Gamma}_i(k) = x_i(0)\left(\frac{1}{n\,\delta_{ii}(k)} - 1\right). \tag{3}$$

The terms $\delta_{ii}(k)$ represent the estimation of the i-th component of the left eigenvector, $\mathbf{w}$, associated to λ_1. The computation of these elements follows the approach in Qu et al. (2012). Each agent handles a vector $\delta_i(k) = [\delta_{i1}(k)\ \ldots\ \delta_{in}(k)]^T$ with initial values $\delta_{ij}(0) = 1$ if $i = j$, and 0 otherwise. The successive values of the vector are computed as $\delta_{ij}(k+1) = \sum_{p\in\mathcal{N}_i\cup\{i\}} \mathcal{C}_{ip}\delta_{pj}(k)$. Defining $\Delta(k) = [\delta_1(k), \ldots, \delta_n(k)]^T$, the previous update can be put in vectorial form using another linear iteration on the matrix $\mathcal{C}$, $\Delta(k+1) = \mathcal{C}\,\Delta(k)$.

Denote

$$\varphi(k) = \mathbf{x}(k) - \mu\mathbf{1}, \tag{4}$$

the disagreement vector of the current estimation with respect to the average of the initial conditions, $\mu = x(0)^T\mathbf{1}/N$. Assuming the weight matrix is diagonalizable, the norm of this vector is bounded by

$$\|\varphi(k)\| \leq \chi_1\, k|\lambda_2|^k + \chi_2|\lambda_2|^k, \tag{5}$$

with $\chi_1,\ \chi_2 \in \mathbb{R}$ two positive constant values and λ_2 the second largest eigenvalue of $\mathcal{C}$, see Proposition 3 in Priolo et al. (2014). In the following, we generalize the result to any row stochastic matrix encoding a SCWD.

3 Convergence Analysis

Let us suppose that the matrix $\mathcal{C}$ has $M \leq N$ distinct eigenvalues, denoted by $\lambda_i,\ i = 1,\ldots,M$. Without loss of generality, let $\mathbf{w}$ be chosen in such a way that $\mathbf{w}^T\mathbf{1} = 1$. The rest of eigenvalues, sorted in modulus, satisfy that $|\lambda_i| < 1, i = 2,\ldots,N$. For each eigenvalue λ_i, we denote by a_i and g_i its algebraic and geometric multiplicity and we define $d_i = a_i - g_i \geq 0$ as their difference. Additionally, we let

$$d_{\max} = \max_i d_i. \tag{6}$$

The main result of this technical communique is the following

Proposition 1 *Let us assume the multi-agent system applies the consensus algorithm give in eq.* (1). *Then, the disagreement vector* $\varphi(k)$ *in eq.* (4) *can be bounded as*

$$\|\varphi(k)\| < \chi_1\, k^{2d_{\max}+1}|\lambda_2|^{k-2d_{\max}} + \chi_2 k^{d_{\max}}\left|\lambda_2\right|^{k-d_{\max}}, \tag{7}$$

with $\|\cdot\|$ *the Euclidean norm,* $d_{\max}$ *defined in* (6) *and* $\chi_1,\ \chi_2 \in \mathbb{R}$ *two positive constant values.*

The rest of the section is devoted to the demonstration of Proposition 1. We begin by introducing several lemmas that provide intermediate bounds. First of all, we provide a bound for the disagreement vector of a linear iteration with respect to the weighted average of the initial conditions given by the left eigenvector.

Lemma 3.1 *Given a vector* $\mathbf{x} \in \mathbb{R}^n$, *for all* $k \in \mathbb{N}$, *it holds that*

$$\|\mathcal{C}^k\mathbf{x} - \mathbf{w}^T\mathbf{x}\mathbf{1}\| \leq \chi k^{d_{\max}}|\lambda_2|^{k-d_{\max}}, \tag{8}$$

with χ *a constant scalar.*

Proof: Let $\mathcal{Q} = \mathcal{C} - \mathbf{1}\mathbf{w}^T$, whose eigenvalues are $\lambda_1 = 0$, with $\mathbf{w}$ and $\mathbf{1}$ its corresponding left and right eigenvectors respectively, while the rest of eigenvalues and eigenvectors are the same as for $\mathcal{C}$. Using the following properties

$$\mathcal{C}^k\left(\mathbf{w}^T\mathbf{x}\right)\mathbf{1} = \left(\mathbf{w}^T\mathbf{x}\right)\mathbf{1},$$
$$\mathbf{1}\mathbf{w}^T(\mathbf{x} - \mathbf{w}^T\mathbf{x}\mathbf{1}) = \mathbf{0},$$

we can see that

$$\mathcal{C}^k(\mathbf{x} - \mathbf{w}^T\mathbf{x}\mathbf{1}) = \mathcal{Q}^k(\mathbf{x} - \mathbf{w}^T\mathbf{x}\mathbf{1}), \tag{9}$$

for all $k \in \mathbb{N}$, and therefore

$$\begin{aligned}\|\mathcal{C}^k\mathbf{x} - \mathbf{w}^T\mathbf{x}\mathbf{1}\| &= \|\mathcal{Q}^k\left(\mathbf{x} - \mathbf{w}^T\mathbf{x}\mathbf{1}\right)\| \\ &\leq \|\mathcal{Q}^k\|\|\mathbf{x} - \mathbf{w}^T\mathbf{x}\mathbf{1}\|.\end{aligned} \tag{10}$$

In order to bound $\|\mathcal{Q}^k\|$, we use the Jordan decomposition

$$\mathcal{Q} = \mathcal{P}\mathcal{J}\mathcal{P}^{-1}, \tag{11}$$

with $\mathcal{J}$ the Jordan matrix, Gantmacher (1990). This matrix is block-diagonal, containing M different blocks, denoted by $\mathcal{J}_i,\ i = 1,\ldots,M$. Each of this blocks can be expressed by the sum

$$\mathcal{J}_i = \lambda_i\mathcal{I}_{a_i} + \mathcal{R}_i, \tag{12}$$

with $\mathcal{I}_{a_i}$ the identity matrix of dimension a_i and $\mathcal{R}_i$ a matrix with all zeros and, if $d_i > 0$, some ones in the elements of the upper-diagonal above the main diagonal. The powers of $\mathcal{Q}$ are equal to $\mathcal{Q}^k = \mathcal{P}\mathcal{J}^k\mathcal{P}^{-1}$. Since $\mathcal{J}$ is block diagonal, we analyze the powers of a particular block. Using (12) and the fact that $\mathcal{R}_i^d = 0$ for all $d > d_i$, the powers of $\mathcal{J}_i$ can be expressed as a sum with the Newton binomial

$$\mathcal{J}_i^k = \sum_{d=0}^{k}\begin{pmatrix}k\\d\end{pmatrix}\lambda_i^{k-d}\mathcal{R}_i^d = \sum_{d=0}^{d_i}\begin{pmatrix}k\\d\end{pmatrix}\lambda_i^{k-d}\mathcal{R}_i^d. \tag{13}$$

Bounding all the binomial numbers by k^{d_i}, and noting that $\|\mathcal{R}_i^d\| \le 1$ for all $d \le d_i$ we have

$$\|\mathcal{J}_i^k\| < (d_i+1)k^{d_i}|\lambda_i|^{k-d_i}. \tag{14}$$

Recalling that $d_{\max} = \max_i d_i$ and bounding the powers of the eigenvalues by the power of the largest eigenvalue among them, $|\lambda_2|^{k-d_i} < 1$, it follows

$$\|\mathcal{J}^k\| < (d_{\max}+1)k^{d_{\max}}|\lambda_2|^{k-d_{\max}}. \tag{15}$$

Replacing in (11)

$$\begin{aligned}\|\mathcal{Q}^k\| &= \|\mathcal{P}\mathcal{J}^k\mathcal{P}^{-1}\| \le \|\mathcal{P}\|\|\mathcal{J}^k\|\|\mathcal{P}^{-1}\| \\ &< \gamma_1(d_{\max}+1)k^{d_{\max}}|\lambda_2|^{k-d_{\max}},\end{aligned} \tag{16}$$

with $\gamma_1 = \|\mathcal{P}\|\|\mathcal{P}^{-1}\|$ a constant. Thus, denoting

$$\chi = \gamma_1(d_{\max}+1)\|\mathbf{x} - \mathbf{w}^T\mathbf{x}\mathbf{1}\|, \tag{17}$$

the bound in (8) follows. ■

Next, we discuss the particular case of the disagreement vector at iteration k caused by the linear iteration of the correction term at iteration $j < k$.

Lemma 3.2 *For all $k \in \mathbb{N}$, and $0 \le j < k$ it holds that*

$$\left\|\mathcal{C}^{k-j}\epsilon(j) - \boldsymbol{w}^T\epsilon(j)\,\boldsymbol{1}\right\| < \chi_1\big((k-j)j\big)^{d_{\max}}|\lambda_2|^{k-2d_{\max}}, \tag{18}$$

with χ_1 a constant scalar.

Proof: Using Lemma 3.1,

$$\left\|\mathcal{C}^{k-j}\epsilon(j) - \mathbf{w}^T\epsilon(j)\,\mathbf{1}\right\| < \chi(k-j)^{d_{\max}}|\lambda_2|^{k-j-d_{\max}},$$

with

$$\chi = \gamma_1(d_{\max}+1)\|\epsilon(j) - \mathbf{w}^T\epsilon(j)\,\mathbf{1}\|. \tag{19}$$

In order to bound the norm of $\epsilon(j) - \mathbf{w}^T\epsilon(j)\,\mathbf{1}$, we analyze its infinity norm

$$\|\epsilon(j) - \mathbf{w}^T\epsilon(j)\,\mathbf{1}\| \le \sqrt{n}\left\|\epsilon(j) - \mathbf{w}^T\epsilon(j)\,\mathbf{1}\right\|_\infty. \tag{20}$$

Combining eqs. (2) and (3) we have

$$\epsilon_i(j) = \frac{x_i(0)}{n}\left(\frac{\delta_{ii}(j-1) - \delta_{ii}(j)}{\delta_{ii}(j)\,\delta_{ii}(j-1)}\right). \tag{21}$$

Therefore, recalling that $\mathbf{w}^T\mathbf{1} = 1$, we obtain

$$\begin{aligned}&\left\|\epsilon(j) - \mathbf{w}^T\epsilon(j)\,\mathbf{1}\right\|_\infty \\ &= \frac{1}{n}\max_i\left|x_i(0)\frac{\delta_{ii}(j-1)-\delta_{ii}(j)}{\delta_{ii}(j)\,\delta_{ii}(j)}\right. \\ &\qquad\left. - \sum_\ell w_\ell x_\ell(0)\frac{\delta_{\ell\ell}(j-1)-\delta_{\ell\ell}(j)}{\delta_{\ell\ell}(j)\,\delta_{\ell\ell}(j)}\right| \\ &= \frac{1}{n}\max_i\left|x_i(0)\frac{\delta_{ii}(j-1)-\delta_{ii}(j)}{\delta_{ii}(j)\,\delta_{ii}(j)}\right| \\ &\qquad + \sum_\ell w_\ell\max_i\left|x_i(0)\frac{\delta_{ii}(j-1)-\delta_{ii}(j)}{\delta_{ii}(j)\,\delta_{ii}(j)}\right| \\ &< \frac{2\max_i|x_i(0)|}{n}\underbrace{\max_i\left|\frac{\delta_{ii}(j-1)-\delta_{ii}(j)}{\delta_{ii}(j)\,\delta_{ii}(j)}\right|}_{\alpha_1}.\end{aligned} \tag{22}$$

The next step is to find a bound for α_1. Since $\delta_{ii}(j)\mathbf{1} = \mathcal{C}^je_i$ with e_i the i-th vector of the canonical basis (see Priolo et al. (2014)), it follows

$$\begin{aligned}\alpha_1 &= \max_i\frac{\left|\delta_{ii}(j-1) - w_i + w_i - \delta_{ii}(j)\right|}{\left|\delta_{ii}(j-1)\,\delta_{ii}(j)\right|} \\ &< \max_i\frac{\left\|\mathcal{C}^je_i - w_i\,\mathbf{1}\right\| + \left\|\mathcal{C}^{j-1}e_i - w_i\,\mathbf{1}\right\|}{\bar{\delta}^2},\end{aligned} \tag{23}$$

where $\bar{\delta} = \min_{i\in\mathcal{V}}\min_{j\in\mathbb{N}}\delta_{ii}(j) > 0$.
Using again Lemma 3.1 and the fact that $\left\|e_i - w_i\,\mathbf{1}\right\| \le \sqrt{2}$ we obtain

$$\left\|\mathcal{C}^je_i - w_i\,\mathbf{1}\right\| < \gamma_1(d_{\max}+1)\sqrt{2}j^{d_{\max}}|\lambda_2|^{j-d_{\max}}. \tag{24}$$

Thus we have

$$\alpha_1 < \frac{2\gamma_1(d_{\max}+1)\sqrt{2}}{|\lambda_2|\bar{\delta}^2}j^{d_{\max}}|\lambda_2|^{j-d_{\max}}. \tag{25}$$

Finally, combining (19), (20), (22), and (25), the bound in (18) is found with

$$\chi_1 = \frac{4\sqrt{2n}\gamma_1^2(d_{\max}+1)^2\max_i|x_i(0)|}{n|\lambda_2|\bar{\delta}^2}. \tag{26}$$

■

The next lemma bounds the sum of the correction terms that are still required to be introduced at iteration k to guarantee convergence to the average.

Lemma 3.3 *For all $k \in \mathbb{N}$, it holds that*

$$\left\|\sum_{j=k}^\infty\boldsymbol{w}^T\epsilon(j)\,\boldsymbol{1}\right\| < \chi_2k^{d_{\max}}|\lambda_2|^{k-d_{\max}}, \tag{27}$$

with χ_2 a constant scalar.

Proof: Following a similar reasoning as in the previous Lemmas, and using eq. (2)-(3) we obtain

$$\begin{aligned}
\Big\|\sum_{j=k}^{\infty}\mathbf{w}^T\epsilon(j)\,\mathbf{1}\Big\| &\leq \sqrt{n}\Big\|\sum_{j=k}^{\infty}\mathbf{w}^T\epsilon(j)\,\mathbf{1}\Big\|_{\infty} \\
&= \sqrt{n}\left|\sum_{j=k}^{\infty}\sum_i w_i\epsilon_i(j)\right| = \sqrt{n}\left|\sum_i w_i\sum_{j=k}^{\infty}\epsilon_i(j)\right| \\
&= \sqrt{n}\left|\sum_i w_i \lim_{j\to\infty}\tilde{\Gamma}_i(j)-\tilde{\Gamma}_i(k-1)\right| \\
&\leq \frac{\sqrt{n}}{n}\max_i|x_i(0)|\Big|\frac{\delta_{ii}(k-1)-w_i}{\delta_{ii}(k-1)}\Big| \\
&< \frac{\sqrt{n}}{n\bar{\delta}}\max_i|x_i(0)|\,\|\mathcal{C}^{k-1}e_i-w_i\mathbf{1}\| \\
&< \frac{\sqrt{2n}\gamma_1(d_{\max}+1)\max_i|x_i(0)|}{n|\lambda_2|\bar{\delta}}k^{d_{\max}}|\lambda_2|^{k-d_{\max}}.
\end{aligned}$$

Therefore, (27) holds with

$$\chi_2 = \frac{\sqrt{2n}\gamma_1(d_{\max}+1)\max_i|x_i(0)|}{n|\lambda_2|\bar{\delta}}. \tag{28}$$

■

Finally, we provide the proof of Proposition 1:

Proof of Proposition 1: The average of the initial conditions can be expressed by

$$\mu = \mathbf{w}^T x(0) + \sum_{k=0}^{\infty}\mathbf{w}^T\epsilon(k). \tag{29}$$

Therefore, using (1), the norm of the disagreement vector can be bounded as

$$\begin{aligned}
\|\varphi(k)\| \leq{} & \Big\|\mathcal{C}^k x(0)-\mathbf{w}^T x(0)\,\mathbf{1}\Big\| \\
&+\Big\|\sum_{j=0}^{k-1}\mathcal{C}^{k-j}\epsilon(j)-\sum_{j=0}^{\infty}\mathbf{w}^T\epsilon(j)\,\mathbf{1}\Big\| \\
\leq{} & \Big\|\mathcal{C}^k x(0)-\mathbf{w}^T x(0)\,\mathbf{1}\Big\| \\
&+\Big\|\sum_{j=0}^{k-1}\Big(\mathcal{C}^{k-j}\epsilon(j)-\mathbf{w}^T\epsilon(j)\,\mathbf{1}\Big)\Big\| \\
&+\Big\|\sum_{j=k}^{\infty}\mathbf{w}^T\epsilon(j)\,\mathbf{1}\Big\|.
\end{aligned} \tag{30}$$

The first term of the right hand side of the inequality is bounded by Lemma 3.1 and the last term is bounded by Lemma 3.3. These two bounds together yield the second term of the right hand side of (7).

The other term of the bound in (7) is found using Lemma 3.2 on the remaining sum as

$$\begin{aligned}
&\Big\|\sum_{j=0}^{k-1}\Big(\mathcal{C}^{k-j}\epsilon(j)-\mathbf{w}^T\epsilon(j)\,\mathbf{1}\Big)\Big\| \\
&\quad\leq \sum_{j=0}^{k-1}\Big\|\Big(\mathcal{C}^{k-j}\epsilon(j)-\mathbf{w}^T\epsilon(j)\,\mathbf{1}\Big)\Big\| \\
&\quad< \chi_1\sum_{j=0}^{k-1}\left((k-j)j\right)^{d_{\max}}|\lambda_2|^{k-2d_{\max}} \\
&\quad< \chi_1 k^{2d_{\max}+1}|\lambda_2|^{k-2d_{\max}}.
\end{aligned} \tag{31}$$

■

We point out that the bounds found in Lemmas 3.1 and 3.3 are tight. Unfortunately, the same does not hold for Lemma 3.2. This can be explained by the fact that in proving this bound we considered the norm of $\|\mathcal{Q}^{k-j}\|$ and $\|\mathcal{Q}^j\|$ separately, which results in a looseness of the bound itself. Nevertheless, we reiterate that this shows the convergence of the algorithm for any row stochastic matrix encoding a SCWD.

To conclude, the following remark establishes a relationship with the analysis carried out in Priolo et al. (2014).

Remark 3.1 *If the matrix $\mathcal{C}$ is diagonalizable, then $d_{\max}=0$, and the bound in eq. (7) reduces to (up to a constant) that in Proposition 3 in Priolo et al. (2014).*

4 Conclusion

This technical communique represents a generalization of the theoretical analysis for the consensus algorithm proposed in Priolo et al. (2014). Specifically, we have provided a convergence analysis of the algorithm for any SCWD not restricted to diagonalizable matrices.

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