The Observability Radius of Networks

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***Abstract*—This paper studies the observability radius of network systems, which measures the robustness of a network to perturbations of the edges. We consider linear networks, where the dynamics are described by a weighted adjacency matrix, and dedicated sensors are positioned at a subset of nodes. We allow for perturbations of certain edge weights, with the objective of preventing observability of some modes of the network dynamics. To comply with the network setting, our work considers perturbations with a desired sparsity structure, thus extending the classic literature on the observability radius of linear systems. The paper proposes two sets of results. First, we propose an optimization framework to determine a perturbation with smallest Frobenius norm that renders a desired mode unobservable from the existing sensor nodes. Second, we study the expected observability radius of networks with given structure and random edge weights. We provide fundamental robustness bounds dependent on the connectivity properties of the network and we analytically characterize optimal perturbations of line and star networks, showing that line networks are inherently more robust than star networks.**

I. Introduction

Networks are broadly used to model engineering, social, and natural systems. An important property of such systems is their robustness to contingencies, including failure of components affecting the flow of information, external disturbances altering individual node dynamics, and variations in the network topology and weights. It remains an outstanding problem to quantify how different topological features enable robustness, and to engineer complex networks that remain operable in the face of arbitrary, and perhaps malicious perturbations.

Observability of a network guarantees the ability to reconstruct the state of each node from sparse measurements. While observability is a binary notion [2], the degree of observability, akin to the degree of controllability, can be quantified in different ways, including the energy associated with the measurements [3], [4], the novelty of the output signal [5], the number of necessary sensor nodes [6], [7], and the robustness to removal of interconnection edges [8]. A quantitative notion of observability is preferable over a binary one, as it allows to compare different observable networks, select optimal sensor nodes, and identify topological features favoring observability.

In this work we measure robustness of a network based on the size of the smallest perturbation needed to prevent observability. Our notion of robustness is motivated by the fact that observability is a generic property [9] and network weights are rarely known without uncertainty. For these reasons numerical tests to assess observability may be unreliable and in fact fail to recognize unobservable systems: instead, our measure of observability robustness can be more reliably evaluated [10]. Among our contributions, we highlight connections between the robustness of a network and its structure, and we propose an algorithmic procedure to construct optimal perturbations. Our work finds applicability in network control problems where the network weights can be changed, in security applications where an attacker gains control of some network edges, and in network science for the classification of edges and the design of robust topologies.

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Related work Our study is inspired by classic works on the observability radius of dynamical systems [11], [12], [13], defined as the norm of the smallest perturbation yielding unobservability or, equivalently, the distance to the nearest unobservable realization. For a linear system described by the pair (A, C), the radius of observability has been classically defined as

$$\mu(A, C) = \min_{\Delta_A, \Delta_C} \left\| \begin{bmatrix} \Delta_A \\ \Delta_C \end{bmatrix} \right\|_2 , \quad \text{s.t. } (A + \Delta_A, C + \Delta_C) \text{ is unobservable.}$$

As a known result [12], the observability radius satisfies

$$\mu(A, C) = \min_s \sigma_n \left(\begin{bmatrix} sI - A \\ C \end{bmatrix} \right),$$

where σ_n denotes the smallest singular value, and $s \in \mathbb{R}$ ($s \in \mathbb{C}$ if complex perturbations are allowed). The optimal perturbations Δ_A and Δ_C are typically full matrices and, to the best of our knowledge, all existing results and procedures are not applicable to the case where the perturbations must satisfy a desired sparsity constraint (e.g., see [14]). This scenario is in fact the relevant one for network systems, where the nonzero entries of the network matrices A and C correspond to existing network edges, and it would be undesirable or unrealistic for a perturbation to modify the interaction of disconnected nodes. An exception is the recent paper [8], where structured perturbations are considered in a controllability problem, yet the discussion is limited to the removal of edges.

We depart from the literature by requiring the perturbation to be real, with a desired sparsity pattern, and confined to the network matrix ($\Delta_C = 0$). Our approach builds on the theory of *total least squares* [15]. With respect to existing results on this topic, our work proposes procedures tailored to networks, fundamental bounds, and insights into the robustness of different network topologies.

Contribution The contribution of this paper is threefold. First, we define a metric of network robustness that captures the resilience of a network system to structural, possibly malicious, perturbations. Our metric evaluates the distance of a network from the set of unobservable networks with the same interconnection structure, and it extends existing works on the observability radius of linear systems.

Second, we formulate a problem to determine optimal perturbations (with smallest Frobenius norm) preventing observability. We show that the problem is not convex, derive optimality conditions, and prove that any optimal solution solves a nonlinear generalized eigenvalue problem. Additionally, we propose a numerical procedure based on the power iteration method to determine (sub)optimal solutions.

Third, we derive a fundamental bound on the expected observability radius for networks with random weights. In particular, we present a class of networks for which the expected observability radius decays to zero as the network cardinality increases. Furthermore, we characterize the robustness of line and star networks. In accordance with recent findings on the role of symmetries for the observability and controllability of networks [16], [17], we demonstrate that line networks are inherently more robust than star networks to perturbations of the edge weights. This analysis shows that our measure of robustness can in fact be used to compare different network topologies and guide the design of robust complex systems.

Because the networks we consider are in fact systems with linear dynamics, our results are generally applicable to linear dynamical systems. Yet, our setup allows for perturbations with a fixed sparsity pattern, which may arise from the organization of a network system.

Paper organization The rest of the paper is organized as follows. Section II contains our network model, the definition of the network observability radius, and some preliminary considerations. Section III describes our method to compute network perturbations with smallest Frobenius norm, our optimization algorithm, and an illustrative example. Our bounds on the observability radius of random networks are in Section IV. Finally, Section V concludes the paper.

II. The Network Observability Radius

Consider a directed graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \{1, \dots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ are the vertex and edge sets, respectively. Let $A = [a_{ij}]$ be the *weighted adjacency matrix* of $\mathcal{G}$, where $a_{ij} \in \mathbb{R}$ denotes the weight associated with the edge $(i,j) \in \mathcal{E}$ (representing flow of information from node j to node i), and $a_{ij} = 0$ whenever $(i,j) \notin \mathcal{E}$. Let e_i denote the i-th canonical vector of dimension n. Let $\mathcal{O} = \{o_1, \dots, o_p\} \subseteq \mathcal{V}$ be the set of *sensor nodes*, and define the network output matrix as $C_\mathcal{O} = \begin{bmatrix} e_{o_1} & \cdots & e_{o_p} \end{bmatrix}^\mathsf{T}$. Let $x_i(t) \in \mathbb{R}$ denote the *state* of node i at time t, and let $x : \mathbb{N}_{\geq 0} \to \mathbb{R}^n$ be the map describing the evolution over time of the network state. The network dynamics are described by the linear discrete-time system

$$x(t+1) = A\,x(t), \text{ and } y(t) = C_\mathcal{O}\,x(t), \qquad (1)$$

where $y : \mathbb{N}_{\geq 0} \to \mathbb{R}^p$ is the output of the sensor nodes $\mathcal{O}$.

In this work we characterize structured network perturbations that prevent observability from the sensor nodes. To this aim, let $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ be the *constraint graph*, and define the set of matrices compatible with $\mathcal{H}$ as

$$\mathcal{A}_\mathcal{H} = \{M \,:\, M \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{V}|}, M_{ij} = 0 \text{ if } (i,j) \notin \mathcal{E}_\mathcal{H}\}.$$

Recall from the eigenvector observability test that the network (1) is observable if and only if there is no right eigenvector of A that lies in the kernel of $C_\mathcal{O}$, that is, $C_\mathcal{O} x \neq 0$ whenever $x \neq 0$, $Ax = \lambda x$, and $\lambda \in \mathbb{C}$ [18]. In this work we consider and study the following optimization problem:

$$\begin{aligned} \min \quad & \|\Delta\|_\text{F}^2, & & \\ \text{s.t.} \quad & (A+\Delta)x = \lambda x, & & \text{(eigenvalue constraint)}, \\ & \|x\|_2 = 1, & & \text{(eigenvector constraint)}, \\ & C_\mathcal{O} x = 0, & & \text{(unobservability)}, \\ & \Delta \in \mathcal{A}_\mathcal{H}, & & \text{(structural constraint)}, \end{aligned} \qquad (2)$$

where the minimization is carried out over the eigenvector $x \in \mathbb{C}^n$, the unobservable eigenvalue $\lambda \in \mathbb{C}$, and the network perturbation $\Delta \in \mathbb{R}^{n \times n}$. The function $\|\cdot\|_\text{F} : \mathbb{R}^{n \times n} \to \mathbb{R}_{\geq 0}$ is the Frobenius norm, and $\mathcal{A}_\mathcal{H}$ expresses the desired sparsity pattern of the perturbation. It should be observed that (i) the minimization problem (2) is not convex because the variables Δ and x are multiplied each other in the eigenvector constraint $(A+\Delta)x = \lambda x$, (ii) if $A \in \mathcal{A}_\mathcal{H}$, then the minimization problem is feasible if and only if there exists a network matrix $A + \Delta = \tilde{A} \in \mathcal{A}_\mathcal{H}$ satisfying the eigenvalue and eigenvector constraint, and (iii) if $\mathcal{H} = \mathcal{G}$, then the perturbation modifies the weights of the existing edges only. We make the following assumption:

(A1) The pair $(A, C_\mathcal{O})$ is observable.

Assumption (A1) implies that the perturbation Δ must be nonzero to satisfy the constraints in (2).

For the pair $(A, C_\mathcal{O})$, the *network observability radius* is the solution to the optimization problem (2), which quantifies the total edge perturbation to achieve unobservability. Different cost functions may be of interest and are left as the subject of future research.

The minimization problem (2) can be solved by two subsequent steps. First, we fix the eigenvalue λ, and compute an optimal perturbation that solves the minimization problem for that λ. This computation is the topic of the next section. Second, we search the complex plane for the optimal λ yielding the perturbation with minimum cost. We observe that (i) the exhaustive search of the optimal λ is an inherent feature of this class of problems, as also highlighted in prior work [13]; (ii) in some cases and for certain network topologies the optimal λ can be found analytically, as we do in Section IV for line and star networks; and (iii) in certain applications the choice of λ is guided by the objective of the network perturbation, such as inducing unobservability of unstable modes.

III. Optimality Conditions and Algorithms for the Network Observability Radius

In this section we consider problem (2) with *fixed* λ. Specifically, we address the following minimization problem: given a constraint graph $\mathcal{H}$, the network matrix $A \in \mathcal{A}_\mathcal{G}$, an output matrix $C_\mathcal{O}$, and a desired unobservable eigenvalue $\lambda \in \mathbb{C}$, determine a perturbation $\Delta^* \in \mathbb{R}^{n \times n}$ satisfying

$$\begin{aligned} \|\Delta^*\|_\text{F}^2 = & \min_{x \in \mathbb{C}^n, \Delta \in \mathbb{R}^{n \times n}} & & \|\Delta\|_\text{F}^2, \\ & \text{s.t.} & & (A+\Delta)x = \lambda x, \\ & & & \|x\|_2 = 1, \\ & & & C_\mathcal{O} x = 0, \\ & & & \Delta \in \mathcal{A}_\mathcal{H}. \end{aligned} \qquad (3)$$

From (3), the value $\|\Delta^*\|_\text{F}^2$ equals the observability radius of the network A with sensor nodes $\mathcal{O}$, constraint graph $\mathcal{H}$, and fixed unobservable eigenvalue λ.

A. Optimal network perturbation

We now shape minimization problem (3) to facilitate its solution. Without affecting generality, relabel the network nodes such that the sensor nodes set satisfy

$$\mathcal{O} = \{1, \dots, p\}, \text{ so that } C_\mathcal{O} = \begin{bmatrix} I_p & 0 \end{bmatrix}. \qquad (4)$$

Accordingly,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \text{ and } \Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix}, \qquad (5)$$

where $A_{11} \in \mathbb{R}^{p \times p}$, $A_{12} \in \mathbb{R}^{p \times n-p}$, $A_{21} \in \mathbb{R}^{n-p \times p}$, and $A_{22} \in \mathbb{R}^{n-p \times n-p}$. Let $V = [v_{ij}]$ be the unweighted adjacency matrix of $\mathcal{H}$, where $v_{ij} = 1$ if $(i,j) \in \mathcal{E}_\mathcal{H}$, and $v_{ij} = 0$ otherwise. Following the partitioning of A in (5), let

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}.$$

We perform the following three simplifying steps.

(1–Rewriting the structural constraints) Let $B = A + \Delta$, and notice that $\|\Delta\|_\text{F}^2 = \sum_{i=1}^n \sum_{j=1}^n (b_{ij} - a_{ij})^2$. Then, the minimization problem (3) can equivalently be rewritten restating the constraint $\Delta \in \mathcal{A}_\mathcal{H}$, as in the following:

$$\|\Delta\|_\text{F}^2 = \|B - A\|_\text{F}^2 = \sum_{i=1}^n \sum_{j=1}^n (b_{ij} - a_{ij})^2 v_{ij}^{-1}.$$

Notice that $\|\Delta\|_\text{F}^2 = \infty$ whenever Δ does not satisfy the structural constraint, that is, when $v_{ij} = 0$ and $b_{ij} \neq a_{ij}$.

(2–Minimization with real variables) Let $\lambda = \lambda_\Re + \mathrm{i}\lambda_\Im$, where i denotes the imaginary unit. Let

$$x_\Re = \begin{bmatrix} x_\Re^1 \\ x_\Re^2 \end{bmatrix}, \text{ and } x_\Im = \begin{bmatrix} x_\Im^1 \\ x_\Im^2 \end{bmatrix},$$

denote the real and imaginary parts of the eigenvector x, with $x_\Re^1 \in \mathbb{R}^p$, $x_\Im^1 \in \mathbb{R}^p$, $x_\Re^2 \in \mathbb{R}^{n-p}$, and $x_\Im^2 \in \mathbb{R}^{n-p}$.

Lemma 3.1: ***(Minimization with real eigenvector constraint)*** The constraint $(A + \Delta)x = \lambda x$ can equivalently be written as

$$\begin{aligned} (A + \Delta - \lambda_\Re I)x_\Re &= -\lambda_\Im x_\Im, \\ (A + \Delta - \lambda_\Re I)x_\Im &= \lambda_\Im x_\Re. \end{aligned} \tag{6}$$

Proof: By considering separately the real and imaginary part of the eigenvalue constraint, we have $(A + \Delta)x = \lambda_\Re x + \mathrm{i}\lambda_\Im x$ and $(A + \Delta)\bar{x} = \lambda_\Re \bar{x} - \mathrm{i}\lambda_\Im \bar{x}$, where $\bar{x}$ denotes the complex conjugate of x. Notice that

$$\underbrace{(A + \Delta)(x + \bar{x})}_{(A+\Delta)2x_\Re} = \underbrace{(\lambda_\Re + \mathrm{i}\lambda_\Im)x + (\lambda_\Re - \mathrm{i}\lambda_\Im)\bar{x}}_{2\lambda_\Re x_\Re - 2\lambda_\Im x_\Im},$$

and, analogously,

$$\underbrace{(A + \Delta)(x - \bar{x})}_{(A+\Delta)2\mathrm{i}x_\Im} = \underbrace{(\lambda_\Re + \mathrm{i}\lambda_\Im)x - (\lambda_\Re - \mathrm{i}\lambda_\Im)\bar{x}}_{2\mathrm{i}\lambda_\Re x_\Im + 2\mathrm{i}\lambda_\Im x_\Re},$$

which concludes the proof. ∎

Thus, the problem (3) can be solved over real variables only.

(3–Reduction of dimensionality) The constraint $C_\mathcal{O} x = 0$ and equation (4) imply that $x_\Re^1 = x_\Im^1 = 0$. Thus, in the minimization problem (5) we set $\Delta_{11} = 0$, $\Delta_{21} = 0$, and consider the minimization variables $x_\Re^2$, $x_\Im^2$, Δ_{12}, and Δ_{22}.

These simplifications lead to the following result.

Lemma 3.2: ***(Equivalent minimization problem)*** Let

$$\begin{aligned} &\bar{A} = \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}, \bar{\Delta} = \begin{bmatrix} \Delta_{12} \\ \Delta_{22} \end{bmatrix}, \bar{M} = \begin{bmatrix} 0_{p \times n-p} \\ \lambda_\Im I_{n-p} \end{bmatrix}, \\ &\bar{N} = \begin{bmatrix} 0_{p \times n-p} \\ \lambda_\Re I_{n-p} \end{bmatrix}, \bar{V} = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix}, \text{ and } \bar{B} = \bar{A} + \bar{\Delta}. \end{aligned} \tag{7}$$

The following minimization problem is equivalent to (3):

$$\begin{aligned} \|\bar{\Delta}^*\|_\mathrm{F}^2 = \min_{\bar{B}, x_\Re^2, x_\Im^2} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n-p} (\bar{b}_{ij} - \bar{a}_{ij})^2 v_{ij}^{-1}, \\ \text{s.t.} \quad & \begin{bmatrix} \bar{B} - \bar{N} & \bar{M} \\ -\bar{M} & \bar{B} - \bar{N} \end{bmatrix} \begin{bmatrix} x_\Re^2 \\ x_\Im^2 \end{bmatrix} = 0, \\ & \left\| \begin{bmatrix} x_\Re^2 \\ x_\Im^2 \end{bmatrix} \right\|_2 = 1. \end{aligned} \tag{8}$$

The minimization problem (8) belongs to the class of *(structured) total least squares* problems, which arise in several estimation and identification problems in control theory and signal processing. Our approach is inspired by [15], with the difference that we focus on real perturbations Δ and complex eigenvalue λ: this constraint leads to different optimality conditions and algorithms. Let $A \otimes B$ denote the Kronecker product between the matrices A and B, and $\mathrm{diag}(d_1, \dots, d_n)$ the diagonal matrix with scalar entries $d_1, \dots, d_n$. We now derive the optimality conditions for the problem (8).

Theorem 3.3: ***(Optimality conditions)*** Let $x_\Re^*$, and $x_\Im^*$ be a solution to the minimization problem (8). Then,

$$\begin{aligned} \underbrace{\begin{bmatrix} \bar{A} - \bar{N} & \bar{M} \\ -\bar{M} & \bar{A} - \bar{N} \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} x_\Re^* \\ x_\Im^* \end{bmatrix}}_{x^*} &= \sigma \underbrace{\begin{bmatrix} S_x & T_x \\ T_x & Q_x \end{bmatrix}}_{D_x} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y^*}, \\ \underbrace{\begin{bmatrix} \bar{A} - \bar{N} & \bar{M} \\ -\bar{M} & \bar{A} - \bar{N} \end{bmatrix}^\mathsf{T}}_{\tilde{A}^\mathsf{T}} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y^*} &= \sigma \underbrace{\begin{bmatrix} S_y & T_y \\ T_y & Q_y \end{bmatrix}}_{D_y} \underbrace{\begin{bmatrix} x_\Re^* \\ x_\Im^* \end{bmatrix}}_{x^*}, \end{aligned} \tag{9}$$

for some $\sigma > 0$ and $y^* \in \mathbb{R}^{2n}$ with $\|y^*\| = 1$, and where

$$\begin{aligned} &D_1 = \mathrm{diag}(v_{11}, \dots, v_{1n}, v_{21}, \dots, v_{2n}, \dots, v_{n1}, \dots, v_{nn}), \\ &D_2 = \mathrm{diag}(v_{11}, \dots, v_{n1}, v_{12}, \dots, v_{n2}, \dots, v_{1n}, \dots, v_{nn}), \\ &S_x = (I \otimes x_\Re^*)^\mathsf{T} D_1 (I \otimes x_\Re^*),\ T_x = (I \otimes x_\Re^*)^\mathsf{T} D_1 (I \otimes x_\Im^*), \\ &Q_x = (I \otimes x_\Im^*)^\mathsf{T} D_1 (I \otimes x_\Im^*),\ S_y = (I \otimes y_1)^\mathsf{T} D_2 (I \otimes y_1), \\ &T_y = (I \otimes y_1)^\mathsf{T} D_2 (I \otimes y_2),\ Q_y = (I \otimes y_2)^\mathsf{T} D_2 (I \otimes y_2). \end{aligned} \tag{10}$$

Proof: We adopt the method of Lagrange multipliers to derive optimality conditions for the problem (8). The Lagrangian is

$$\begin{aligned} \mathcal{L}(\bar{B}, x_\Re^2, x_\Im^2, \ell_1, \ell_2, \rho) &= \sum_i \sum_j (\bar{b}_{ij} - \bar{a}_{ij})^2 v_{ij}^{-1} \\ &+ \ell_1^\mathsf{T}((\bar{B} - \bar{N})x_\Re^2 + \bar{M}x_\Im^2) + \ell_2^\mathsf{T}((\bar{B} - \bar{N})x_\Im^2 - \bar{M}x_\Re^2) \\ &+ \rho(1 - x_\Re^{2\mathsf{T}} x_\Re^2 - x_\Im^{2\mathsf{T}} x_\Im^2), \end{aligned} \tag{11}$$

where $\ell_1 \in \mathbb{R}^n$, $\ell_2 \in \mathbb{R}^n$, and $\rho \in \mathbb{R}$ are Lagrange multipliers. By equating the partial derivatives of $\mathcal{L}$ to zero we obtain

$$\frac{\partial \mathcal{L}}{\partial b_{ij}} = 0 \Rightarrow -2(\bar{a}_{ij} - \bar{b}_{ij})v_{ij}^{-1} + \ell_{1i} x_{\Re j}^2 + \ell_{2i} x_{\Im j}^2 = 0, \tag{12}$$

$$\frac{\partial \mathcal{L}}{\partial x_\Re^2} = 0 \Rightarrow \ell_1^\mathsf{T}(\bar{B} - \bar{N}) - \ell_2^\mathsf{T}\bar{M} - 2\rho x_\Re^{2\mathsf{T}} = 0, \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial x_\Im^2} = 0 \Rightarrow \ell_1^\mathsf{T}\bar{M} + \ell_2^\mathsf{T}(\bar{B} - \bar{N}) - 2\rho x_\Im^{2\mathsf{T}} = 0, \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial \ell_1} = 0 \Rightarrow (\bar{B} - \bar{N})x_\Re^2 + \bar{M}x_\Im^2 = 0, \tag{15}$$

$$\frac{\partial \mathcal{L}}{\partial \ell_2} = 0 \Rightarrow (\bar{B} - \bar{N})x_\Im^2 - \bar{M}x_\Re^2 = 0, \tag{16}$$

$$\frac{\partial \mathcal{L}}{\partial \rho} = 0 \Rightarrow x_\Re^{2\mathsf{T}} x_\Re^2 + x_\Im^{2\mathsf{T}} x_\Im^2 = 1. \tag{17}$$

Let $L_1 = \mathrm{diag}(\ell_1)$, $L_2 = \mathrm{diag}(\ell_2)$, $X_\Re = \mathrm{diag}(x_\Re^2)$, $X_\Im = \mathrm{diag}(x_\Im^2)$. After including the factor 2 into the multipliers, equation (12) can be written in matrix form as

$$\bar{A} - \bar{B} = L_1 \bar{V} X_\Re + L_2 \bar{V} X_\Im. \tag{18}$$

Analogously, equations (13) and (14) can be written as

$$\begin{bmatrix} \ell_1^\mathsf{T} & \ell_2^\mathsf{T} \end{bmatrix} \begin{bmatrix} \bar{B} - \bar{N} & \bar{M} \\ -\bar{M} & \bar{B} - \bar{N} \end{bmatrix} - 2\rho \begin{bmatrix} x_\Re^{2\mathsf{T}} & x_\Im^{2\mathsf{T}} \end{bmatrix} = 0, \tag{19}$$

From equation (19) we have

$$\begin{bmatrix} \ell_1^\mathsf{T} & \ell_2^\mathsf{T} \end{bmatrix} \underbrace{\begin{bmatrix} \bar{B} - \bar{N} & \bar{M} \\ -\bar{M} & \bar{B} - \bar{N} \end{bmatrix} \begin{bmatrix} x_\Re^2 \\ x_\Im^2 \end{bmatrix}}_{=0 \text{ due to (15) and (16)}} - 2\rho = 0,$$

from which we conclude $\rho = 0$. By combining (15) and (18) (respectively, (16) and (18)) we obtain

$$\begin{aligned} (\bar{A} - \bar{N})x_\Re^2 + \bar{M}x_\Im^2 &= \left(L_1 \bar{V} X_\Re + L_2 \bar{V} X_\Im\right) x_\Re^2, \\ (\bar{A} - \bar{N})x_\Im^2 - \bar{M}x_\Re^2 &= \left(L_1 \bar{V} X_\Re + L_2 \bar{V} X_\Im\right) x_\Im^2. \end{aligned}$$

Analogously, by combining (13) and (18), (14) and (18), we obtain

$$\begin{aligned} \ell_1^\mathsf{T}(\bar{A} - \bar{N}) - \ell_2^\mathsf{T}\bar{M} &= \ell_1^\mathsf{T} \left(L_1 \bar{V} X_\Re + L_2 \bar{V} X_\Im\right), \\ \ell_2^\mathsf{T}(\bar{A} - \bar{N}) + \ell_1^\mathsf{T}\bar{M} &= \ell_2^\mathsf{T} \left(L_1 \bar{V} X_\Re + L_2 \bar{V} X_\Im\right). \end{aligned}$$

Let $\sigma = \sqrt{\ell_1^\mathsf{T}\ell_1 + \ell_2^\mathsf{T}\ell_2}$ and observe that σ cannot be zero. Indeed, due to Assumption (A1), the optimal perturbation can not be zero; thus, the first constraint in (8) must be active and the corresponding multiplier must be nonzero. Then, we can define $y_1 = \ell_1/\sigma$ and $y_2 = \ell_2/\sigma$ and we can verify that

$$\begin{aligned} \left(L_1 \bar{V} X_\Re + L_2 \bar{V} X_\Im\right) x_\Re^2 &= \sigma \left(S_x y_1 + T_x y_2\right), \\ \left(L_1 \bar{V} X_\Re + L_2 \bar{V} X_\Im\right) x_\Im^2 &= \sigma \left(T_x y_1 + Q_x y_2\right), \end{aligned}$$

and

$$\begin{aligned}\sigma\left(y_1^\mathsf{T}(\bar{A}-\bar{N})-y_2^\mathsf{T}\bar{M}\right) &= \ell_1^\mathsf{T}\left(L_1\bar{V}X_\Re+L_2\bar{V}X_\Im\right)\\ &= \sigma^2\left(S_yx_\Re^2+T_yx_\Im^2\right)^\mathsf{T},\\ \sigma\left(y_2^\mathsf{T}(\bar{A}-\bar{N})+y_1^\mathsf{T}\bar{M}\right) &= \ell_2^\mathsf{T}\left(L_1\bar{V}X_\Re+L_2\bar{V}X_\Im\right)\\ &= \sigma^2\left(T_yx_\Re^2+Q_yx_\Im^2\right)^\mathsf{T},\end{aligned}$$

which conclude the proof. ■

Note that equations (9) may admit multiple solutions, and that every solution to (9) yields a network perturbation that satisfies the constraints in the minimization problem (8). We now present the following result to compute perturbations.

Corollary 3.4: ***(Minimum norm perturbation)*** Let Δ^* be a solution to (3). Then, $\Delta^* = [0^{n\times p}\ \bar{\Delta}^*]$, where

$$\bar{\Delta}^* = -\sigma\left(\operatorname{diag}(y_1)\bar{V}\operatorname{diag}(x_\Re^*)-\operatorname{diag}(y_2)\bar{V}\operatorname{diag}(x_\Im^*)\right),$$

and $x_\Re^*$, $x_\Im^*$, y_1, y_2, σ satisfy the equations (9). Moreover,

$$\|\Delta\|_\text{F}^2 = \sigma^2 x^{*\mathsf{T}}D_yx^* = \sigma x^{*\mathsf{T}}\tilde{A}^\mathsf{T}y^* \leq \sigma\|\tilde{A}\|_\text{F}.$$

Proof: The expression for the perturbation Δ^* comes from Lemma 3.2 and (18), and the fact that $L_1 = \sigma\operatorname{diag}(y_1)$, $L_2 = \sigma\operatorname{diag}(y_2)$. To show the second part notice that

$$\begin{aligned}\|\Delta\|_\text{F}^2 &= \|A-B\|_\text{F}^2 = \|L_1\bar{V}X_\Re+L_2\bar{V}X_\Im\|_\text{F}^2\\ &= \sigma^2\sum_i\sum_j\left(y_{1i}^2x_{\Re j}^2+y_{2i}^2x_{\Im j}^2\right)v_{ij}\\ &= \sigma^2x^{*\mathsf{T}}D_yx^* = \sigma x^{*\mathsf{T}}\tilde{A}^\mathsf{T}y^*,\end{aligned}$$

where the last equalities follow from (9). Finally, the inequality follows from $\|x^*\|_2 = \|x^*\|_\text{F} = \|y^*\|_2 = \|y^*\|_\text{F} = 1$. ■

To compute a triple (σ, x^*, y^*) satisfying the condition in Theorem 3.3, observe that (9) can be written in matrix form as

$$\underbrace{\begin{bmatrix}0 & \tilde{A}^\mathsf{T}\\ \tilde{A} & 0\end{bmatrix}}_{H}\underbrace{\begin{bmatrix}x\\ y\end{bmatrix}}_{z} = \bar{\sigma}\underbrace{\begin{bmatrix}D_y & 0\\ 0 & D_x\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}x\\ y\end{bmatrix}}_{z}. \tag{20}$$

Lemma 3.5: ***(Equivalence between Theorem 3.3 and** (20)**)*** Let (σ, x, y), with $x \neq 0$, solve (20). Then, $\sigma \neq 0$ and $y \neq 0$, and the triple $((\alpha\beta)^{-1}\sigma, \alpha x, \beta y)$, with $\alpha = \operatorname{sgn}(\sigma)\|x\|^{-1}$ and $\beta = \|y\|^{-1}$, satisfies the conditions in Theorem 3.3.

Proof: Because $x \neq 0$ and $\tilde{A}$ has full column rank due to Assumption (A1), it follows $\sigma \neq 0$ and $y \neq 0$. Let D_x and D_y be as in (9). Notice that $D_{\alpha x} = \alpha^2 D_x$ and $D_{\beta y} = \beta^2 D_y$. Notice that $(\alpha\beta)^{-1}\sigma > 0$. We have

$$\begin{aligned}\tilde{A}\alpha x &= \frac{\sigma}{\alpha\beta}\alpha^2 D_x\beta y = \alpha\sigma D_x y,\\ \tilde{A}^\mathsf{T}\beta y &= \frac{\sigma}{\alpha\beta}\beta^2 D_y\alpha x = \beta\sigma D_y x,\end{aligned}$$

which concludes the proof. ■

Lemma 3.5 shows that a (sub)optimal network perturbation can in fact be constructed by solving equations (20). It should be observed that, if the matrices S_x, T_x, Q_x, S_y, T_y, and Q_y were constant, then (20) would describe a generalized eigenvalue problem, thus a solution $(\bar{\sigma}, z)$ would be a pair of generalized eigenvalue and eigenvector. These facts will be exploited in the next section to develop a heuristic algorithm to compute a (sub)optimal network perturbation.

Remark 1: ***(Smallest network perturbation with respect to the unobservable eigenvalue)*** In the minimization problem (3) the size of the perturbation Δ^* depends on the desired eigenvalue λ, and it may be of interest to characterize the unobservable eigenvalue $\lambda^* = \lambda_\Re^* + \mathrm{i}\lambda_\Im^*$ yielding the smallest network perturbation that prevents observability. To this aim, we equate to zero the derivatives of the Lagrangian (11) with respect to $\lambda_\Re$ and $\lambda_\Im$ to obtain

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\lambda_\Re} &= 0 \Rightarrow \ell_1^\mathsf{T}\begin{bmatrix}0_p\\ x_\Re^2\end{bmatrix}+\ell_2^\mathsf{T}\begin{bmatrix}0_p\\ x_\Im^2\end{bmatrix} = 0,\\ \frac{\partial\mathcal{L}}{\partial\lambda_\Im} &= 0 \Rightarrow \ell_1^\mathsf{T}\begin{bmatrix}0_p\\ x_\Im^2\end{bmatrix}-\ell_2^\mathsf{T}\begin{bmatrix}0_p\\ x_\Re^2\end{bmatrix} = 0.\end{aligned}$$

The above conditions clarify that, for the perturbation Δ to be of the smallest size with respect to λ, the Lagrange multipliers ℓ_1 and ℓ_2, and the vectors $x_\Re^2$ and $x_\Im^2$ must verify an orthogonality condition. □

Remark 2: ***(Real unobservable eigenvalue)*** When the unobservable eigenvalue λ in (3) is real, the optimality conditions in Theorem 3.3 can be simplified to

$$(\bar{A}-\bar{N})x_\Re = \sigma S_x y_1, \text{ and } (\bar{A}-\bar{N})y_1 = \sigma S_y x_\Re.$$

The generalized eigenvalue equation (20) becomes

$$\begin{bmatrix}0 & \bar{A}^\mathsf{T}-\bar{N}^\mathsf{T}\\ \bar{A}-\bar{N} & 0\end{bmatrix}\begin{bmatrix}y_1\\ x_\Re\end{bmatrix} = \sigma\begin{bmatrix}S_x & 0\\ 0 & S_y\end{bmatrix}\begin{bmatrix}y_1\\ x_\Re\end{bmatrix},$$

and the optimality conditions with respect to the unobservable eigenvalue λ (see Remark 1) simplify to $\ell_1^\mathsf{T}\begin{bmatrix}0_p\\ x_\Re^2\end{bmatrix} = 0$. □

B. A heuristic procedure to compute structural perturbations

In this section we propose an algorithm to find a solution to the set of nonlinear equations (20), and thus to find a (sub)optimal solution to the minimization problem (3). Our procedure is motivated by (20) and Corollary 3.4, and it consists of fixing a vector z, computing the matrix D, and approximating an eigenvector associated with the smallest generalized eigenvalue of the pair (H, D). Because the size of the perturbation is bounded by the generalized eigenvalue σ as in Corollary 3.4, we adopt an iterative procedure based on the *inverse iteration* method for the computation of the smallest eigenvalue of a matrix [19]. We remark that our procedure is heuristic, because (20) is in fact a nonlinear generalized eigenvalue problem due to the dependency of the matrix D on the eigenvector z. To the best of our knowledge, no complete algorithm is known for the solution of (20). We start by characterizing certain properties of the matrices H and D, which will be used to derive our algorithm. Let

$$\operatorname{spec}(H, D) = \{\lambda \in \mathbb{C} \ :\ \det(H-\lambda D) = 0\},$$

and recall that the pencil (H, D) is regular if the determinant $\det(H-\lambda D)$ does not vanish for some value of λ, see [20]. Notice that, if (H, D) is not regular, then $\operatorname{spec}(H, D) = \mathbb{C}$.

Lemma 3.6: ***(Generalized eigenvalues of** (H, D)**)*** Given a vector $z \in \mathbb{R}^{4n-2p}$, define the matrices H and D as in (20). Then,

(i) $0 \in \operatorname{spec}(H, D)$;
(ii) if $\lambda \in \operatorname{spec}(H, D)$, then $-\lambda \in \operatorname{spec}(H, D)$; and
(iii) if (H, D) is regular, then $\operatorname{spec}(H, D) \subset \mathbb{R}$.

Proof: Statement (i) is equivalent to $\tilde{A}x = 0$ and $\tilde{A}^\mathsf{T}y = 0$, for some vectors x and y. Because $\tilde{A}^\mathsf{T} \in \mathbb{R}^{(2n-2p)\times 2n}$ with $p \geq 1$, the matrix $\tilde{A}^\mathsf{T}$ features a nontrivial null space. Thus, the two equations are satisfied with $x = 0$ and $y \in \operatorname{Ker}(\tilde{A}^\mathsf{T})$, and the statement follows.

To prove statement (ii) notice that, due to the block structure of H and D, if the triple $(\lambda, \bar{x}, \bar{y})$ satisfies the generalized eigenvalue equations $\tilde{A}^\mathsf{T}\bar{y} = \lambda D_y\bar{x}$ and $\tilde{A}\bar{x} = \lambda D_x\bar{y}$, so does $(-\lambda, \bar{x}, -\bar{y})$.

To show statement (iii), let $\operatorname{Rank}(D) = k \leq n$, and notice that the regularity of the pencil (H, D) implies $H\bar{z} \neq 0$ whenever $D\bar{z} = 0$ and $\bar{z} \neq 0$. Notice that (H, D) has $n-k$ infinite eigenvalues [20] because $H\bar{z} = \lambda D\bar{z} = \lambda \cdot 0$ for every nontrivial $\bar{z} \in \operatorname{Ker}(D)$.

Because D is symmetric, it admits an orthonormal basis of eigenvectors. Let $V_1 \in \mathbb{R}^{n\times k}$ contain the orthonormal eigenvectors of D associated with its nonzero eigenvalues, let Λ_D be the corresponding diagonal matrix of the eigenvalues, and let $T_1 = V_1\Lambda_D^{-1/2}$. Then, $T_1^\mathsf{T} D T_1 = I$. Let $\tilde{H} = T_1^\mathsf{T} H T_1$, and notice that $\tilde{H}$ is symmetric. Let $T_2 \in \mathbb{R}^{k\times k}$ be an orthonormal matrix of the eigenvectors of $\tilde{H}$. Let $T = T_1T_2$ and note that $T^\mathsf{T} H T = \Lambda$ and $T^\mathsf{T} D T = I$, where Λ is a diagonal matrix. To conclude, consider the generalized eigenvalue problem $H\bar{z} = \lambda D\bar{z}$. Let $\bar{z} = T\tilde{z}$. Because T has full column rank k, we have $T^\mathsf{T} H T\tilde{z} = \Lambda\tilde{z} = \lambda T^\mathsf{T} D T\tilde{z} = \lambda\tilde{z}$, from which we conclude that (H, D) has k real eigenvalues. ■

Lemma 3.6 implies that the inverse iteration method is not directly applicable to (20). In fact, the zero eigenvalue of (H, D) leads the inverse iteration to instability, while the presence of eigenvalues of (H, D) with equal magnitude may induce non-decaying oscillations in the solution vector. To overcome these issues, we employ a shifting mechanism as detailed in Algorithm 1, where the eigenvector z is iteratively updated by solving the equation $(H - \mu D)z_{k+1} = Dz_k$ until a convergence criteria is met. Notice that (i) the eigenvalues of $(H - \mu D, D)$ are shifted with respect to the eigenvalues of (H, D), that is, if $\sigma \in \operatorname{spec}(H, D)$, then $\sigma - \mu \in \operatorname{spec}(H - \mu D, D)$,[1] (ii) the pairs $(H - \mu D, D)$ and (H, D) share the same eigenvectors, and (iii) by selecting $\mu = \psi \cdot \min\{\sigma \in \operatorname{spec}(H, D) \ : \ \sigma > 0\}$, the pair $(H - \mu D, D)$ has nonzero eigenvalues with distinct magnitude. Thus, Algorithm 1 estimates the eigenvector z associated with the smallest nonzero eigenvalue σ of (H, D), and converges when z and σ also satisfy equations (20). The parameter ψ determines a compromise between numerical stability and convergence speed; larger values of ψ improve the convergence speed.[2]

Algorithm 1: Heuristic solution to (20)

Input: Matrix H; max iterations $\max_{\text{iter}}$; $\psi \in (0.5, 1)$.
Output: $(\sigma,\ z)$ satisfying (20), or fail.
repeat
  $z \leftarrow (H - \mu D)^{-1}Dz$;
  $\phi \leftarrow \|z\|$;
  $z \leftarrow z/\phi$;
  $\mu = \psi \cdot \min\{\phi \in \operatorname{spec}(H, D) \ : \ \phi > 0\}$;
  update D according to (10);
  $i \leftarrow i + 1$
until *convergence or* $i > \max_{\text{iter}}$;
return $(\phi + \mu, z)$ *or fail if* $i = \max_{\text{iter}}$;

When convergent, Algorithm 1 finds a solution to (20) and, consequently, the algorithm could stop at a local minimum and return a (sub)optimal network perturbation preventing observability of a desired eigenvalue. All information about the network matrix, the sensor nodes, the constraint graph, and the unobservable eigenvalue is encoded in the matrix H as in (7), (9) and (20). Although convergence of Algorithm 1 is not guaranteed, numerical studies show that it performs well in practice; see Sections III-C and IV.

C. Optimal perturbations and algorithm validation

In this section we validate Algorithm 1 on a small network. We start with the following result.

Theorem 3.7: ***(Optimal perturbations of 3-dimensional line networks with fixed*** $\lambda \in \mathbb{C}$***)*** Consider a network with graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $|\mathcal{V}| = 3$, weighted adjacency matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix},$$

and sensor node $\mathcal{O} = \{1\}$. Let $B = [b_{ij}] = A + \Delta^*$, where Δ^* solves the minimization problem (3) with constraint graph $\mathcal{H} = \mathcal{G}$ and unobservable eigenvalue $\lambda = \lambda_\Re + \mathrm{i}\lambda_\Im \in \mathbb{C}$, $\lambda_\Im \neq 0$. Then:

$$b_{11} = a_{11}, \quad b_{21} = a_{21}, \quad b_{12} = 0,$$

and b_{22}, b_{23}, b_{32}, and b_{33} satisfy:

$$\begin{aligned} (b_{22} - a_{22}) - (b_{33} - a_{33}) + \frac{b_{33} - b_{22}}{b_{32}}(b_{23} - a_{23}) &= 0, \\ (b_{32} - a_{32}) - \frac{b_{23}}{b_{32}}(b_{23} - a_{23}) &= 0, \\ b_{22} + b_{33} - 2\lambda_\Re &= 0, \\ b_{22}b_{33} - b_{23}b_{32} - \lambda_\Re^2 - \lambda_\Im^2 &= 0. \end{aligned} \tag{21}$$

Proof: Let $Bx = \lambda x$ and notice that, because λ is unobservable, $C_\mathcal{O}x = [1\ 0\ 0]x = 0$. Then, $x = [x_1\ x_2\ x_3]^\mathsf{T}$, $x_1 = 0$, $b_{11} = a_{11}$, and $b_{21} = a_{21}$. By contradiction, let $x_2 = 0$. Notice that $Bx = \lambda x$ implies $b_{33} = \lambda$, which contradicts the assumption that $\lambda_\Im \neq 0$ and $b_{33} \in \mathbb{R}$. Thus, $x_2 \neq 0$. Because $x_2 \neq 0$, the relation $Bx = \lambda x$ and $x_1 = 0$ imply $b_{12} = 0$. Additionally, λ is an eigenvalue of

$$B_2 = \begin{bmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{bmatrix}.$$

The characteristic polynomial of B_2 is

$$P_{B_2}(s) = s^2 - (b_{22} + b_{33})s + b_{22}b_{33} - b_{23}b_{32}.$$

For $\lambda \in \operatorname{spec}(B_2)$, we must have $P_{B_2}(s) = (s - \lambda)(s - \bar{\lambda})$, where $\bar{\lambda}$ is the complex conjugate of λ. Thus,

$$P_{B_2}(s) = (s - \lambda_\Re - \mathrm{i}\lambda_\Im)(s - \lambda_\Re + \mathrm{i}\lambda_\Im) = s^2 - 2\lambda_\Re s + \lambda_\Re^2 + \lambda_\Im^2,$$

which leads to

$$b_{22} + b_{33} - 2\lambda_\Re = 0, \text{ and } b_{22}b_{33} - b_{23}b_{32} - \lambda_\Re^2 - \lambda_\Im^2 = 0. \tag{22}$$

The Lagrange function of the minimization problem with cost function $\|\Delta^*\|_\text{F}^2 = \sum_{i=2}^3\sum_{j=2}^3 (b_{ij} - a_{ij})^2$ and constraints (22) is

$$\begin{aligned} \mathcal{L}(b_{22}, b_{23}, b_{32}, b_{33}, p_1, p_2) &= d_{22}^2 + d_{23}^2 + d_{32}^2 + d_{33}^2 \\ &+ p_1(2\lambda_\Re + b_{22} + b_{33}) + p_2(b_{22}b_{33} - b_{23}b_{32} - (\lambda_\Re^2 + \lambda_\Im^2)), \end{aligned}$$

where $p_1, p_2 \in \mathbb{R}$ are Lagrange multipliers, and $d_{ij} = b_{ij} - a_{ij}$. By equating the partial derivatives of $\mathcal{L}$ to zero we obtain

$$\frac{\partial \mathcal{L}}{\partial b_{22}} = 0 \Rightarrow 2d_{22} + p_1 + p_2 b_{33} = 0, \tag{23}$$

$$\frac{\partial \mathcal{L}}{\partial b_{33}} = 0 \Rightarrow 2d_{33} + p_1 + p_2 b_{22} = 0, \tag{24}$$

$$\frac{\partial \mathcal{L}}{\partial b_{23}} = 0 \Rightarrow 2d_{23} - p_2 b_{32} = 0, \tag{25}$$

$$\frac{\partial \mathcal{L}}{\partial b_{32}} = 0 \Rightarrow 2d_{32} - p_2 b_{23} = 0, \tag{26}$$

together with (22). The statement follows by substituting the Lagrange multipliers p_1 and p_2 into (23) and (26). ■

To validate Algorithm 1, in Fig. 1 we compute optimal perturbations for 3-dimensional line networks based on Theorem 3.7, and compare them with the perturbation obtained at with Algorithm 1.

[1] To see this, let σ be an eigenvalue of (H, D), that is, $Hx = \sigma Dx$. Then, $(H - \mu D)x = Hx - \mu Dx = \sigma Dx - \mu Dx = (\sigma - \mu)Dx$. That is $(H - \mu D)x = (\sigma - \mu)Dx$ thus $\sigma - \mu$ is an eigenvalue of $(H - \mu D, D)$.

[2] In Algorithm 1 the range for ψ has been empirically determined during our numerical studies.

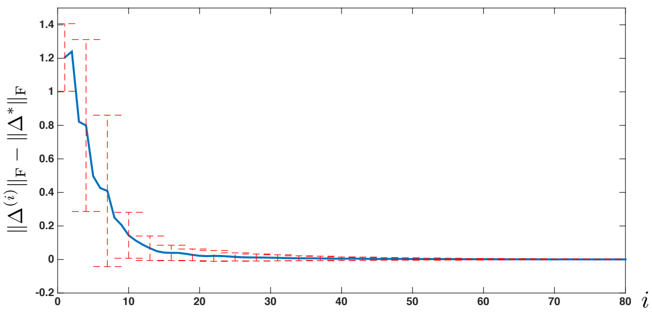


Fig. 1. This figure validates the effectiveness of Algorithm 1 to compute optimal perturbations for the line network in Section III-C. The plot shows the mean and standard deviation over 100 networks of the difference between Δ^*, obtained via the optimality conditions (21), and $\Delta^{(i)}$, computed at the i-th iteration of Algorithm 1. The unobservable eigenvalue is $\lambda = \mathrm{i}$ and the values a_{ij} are chosen independently and uniformly distributed in $[0, 1]$.

IV. Observability Radius of Random Networks: the Case of Line and Star Networks

In this section we study the observability radius of networks with fixed structure and random weights, when the desired unobservable eigenvalue is an optimization parameter as in (2). First, we give a general upper bound on the size of an optimal perturbation. Next, we explicitly compute optimal perturbations for line and star networks, showing that their robustness is essentially different.

We start with some necessary definitions. Given a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a cut is a subset of edges $\bar{\mathcal{E}} \subseteq \mathcal{E}$. Given two disjoint sets of vertices $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{V}$, we say that a cut $\bar{\mathcal{E}}$ disconnects $\mathcal{S}_2$ from $\mathcal{S}_1$ if there exists no path from any vertex in $\mathcal{S}_2$ to any vertex in $\mathcal{S}_1$ in the subgraph $(\mathcal{V}, \mathcal{E} \setminus \bar{\mathcal{E}})$. Two cuts $\mathcal{E}_1$ and $\mathcal{E}_2$ are disjoint if they have no edge in common, that is, if $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$. Finally, the Gamma function is defined as $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, \mathrm{d}x$. With this notation in place, we are in the position to prove a general upper bound on the (expected) norm of the smallest perturbation that prevents observability. The proof is based on the following intuition: a perturbation that disconnects the graph prevents observability.

Theorem 4.1: ***(Bound on expected network observability radius)*** Consider a network with graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, weighted adjacency matrix $A = [a_{ij}]$, and sensor nodes $\mathcal{O} \subseteq \mathcal{V}$. Let the weights a_{ij} be independent random variables uniformly distributed in the interval $[0, 1]$. Define the minimal observability-preventing perturbation as

$$\delta = \min_{\lambda \in \mathbb{C}, x \in \mathbb{C}^n, \Delta \in \mathbb{R}^{n \times n}} \|\Delta\|_F, \tag{27}$$

$$\begin{aligned} \text{s.t.} \quad & (A + \Delta)x = \lambda x, \\ & \|x\|_2 = 1, \\ & C_{\mathcal{O}} x = 0, \\ & \Delta \in \mathcal{A}_{\mathcal{G}}. \end{aligned}$$

Let $\Omega_k(\mathcal{O})$ be a collection of disjoint cuts of cardinality k, where each cut disconnects a non-empty subset of nodes from $\mathcal{O}$. Let $\omega = |\Omega_k(\mathcal{O})|$ be the cardinality of $\Omega_k(\mathcal{O})$. Then,

$$\mathbb{E}[\delta] \leq \frac{\Gamma(1/k)\, \Gamma(\omega + 1)}{\sqrt{k}\, \Gamma(\omega + 1 + 1/k)}.$$

Proof: Let $\bar{\mathcal{E}} \in \Omega_k(\mathcal{O})$. Notice that, after removing the edges $\bar{\mathcal{E}}$, the nodes are partitioned as $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, where $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, $\mathcal{O} \subseteq \mathcal{V}_1$, and $\mathcal{V}_2$ is disconnected from $\mathcal{V}_1$. Reorder the network nodes so that $\mathcal{V}_1 = \{1, \dots, |\mathcal{V}_1|\}$ and $\mathcal{V}_2 = \{|\mathcal{V}_1| + 1, \dots, |\mathcal{V}|\}$. Accordingly, the modified network matrix is reducible and reads as

$$\bar{A} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}.$$

Let x_2 be an eigenvector of A_{22} with corresponding eigenvalue λ. Notice that λ is an eigenvalue of $\bar{A}$ with eigenvector $x = [0 \; x_2^\mathsf{T}]^\mathsf{T}$. Since $\mathcal{O} \subseteq \mathcal{V}_1$, $C_{\mathcal{O}} x = 0$, so that the eigenvalue λ is unobservable.

From the above discussion we conclude that, for each $\bar{\mathcal{E}} \in \Omega_k(\mathcal{O})$, there exists a perturbation $\Delta = [\delta_{ij}]$ that is compatible with $\mathcal{G}$ and ensures that one eigenvalue is unobservable. Moreover, the perturbation Δ is defined as $\delta_{ij} = -a_{ij}$ if $(i, j) \in \bar{\mathcal{E}}$, and $\delta_{ij} = 0$ otherwise. We thus have

$$\mathbb{E}[\delta] \leq \mathbb{E}\left[\min_{\bar{\mathcal{E}} \in \Omega_k(\mathcal{O})} \sqrt{\sum_{(i,j) \in \bar{\mathcal{E}}} a_{ij}^2} \right].$$

Because any two elements of $\Omega_k(\mathcal{O})$ have empty intersection and all edge weights are independent, we have

$$\begin{aligned} \Pr\left(\min_{\bar{\mathcal{E}} \in \Omega_k(\mathcal{O})} \sqrt{\sum_{(i,j) \in \bar{\mathcal{E}}} a_{ij}^2} \geq x \right) &= \Pr\left(\sqrt{\sum_{(i,j) \in \bar{\mathcal{E}}} a_{ij}^2} \geq x \right)^\omega \\ = \Pr\left(\sum_{(i,j) \in \bar{\mathcal{E}}} a_{ij}^2 \geq x^2 \right)^\omega &= \left(1 - \Pr\left(\sum_{(i,j) \in \bar{\mathcal{E}}} a_{ij}^2 \leq x^2 \right) \right)^\omega. \end{aligned}$$

In order to obtain a more explicit expression for this probability, we resort to using a lower bound. Let a denote the vector of a_{ij} with $(i, j) \in \bar{\mathcal{E}}$. The condition $\sum_{(i,j) \in \bar{\mathcal{E}}} a_{ij}^2 \leq x^2$ implies that a belongs to the k-dimensional sphere of radius x (centered at the origin). In fact, since a is sampled in $[0, 1]^k$, it belongs to the intersection between the sphere and the first orthant. By computing the volume of the k-dimensional cube inscribed in the sphere, we obtain

$$\Pr\left(\sum_{(i,j) \in \bar{\mathcal{E}}} a_{ij}^2 \leq x^2 \right) \geq \begin{cases} \frac{(2x/\sqrt{k})^k}{2^k} = \left(\frac{x}{\sqrt{k}} \right)^k, & x \leq \sqrt{k}, \\ 1, & \text{otherwise}. \end{cases}$$

Since δ takes on nonnegative values only, its expectation can be computed by integrating the survival function

$$\mathbb{E}[\delta] = \int_0^\infty \Pr(\delta \geq t)\, \mathrm{d}t,$$

which leads us to obtain, by suitable changes of variables,

$$\begin{aligned} \mathbb{E}[\delta] &\leq \int_0^{\sqrt{k}} \left(1 - \left(\frac{x}{\sqrt{k}} \right)^k \right)^\omega \mathrm{d}x = \sqrt{k} \int_0^1 \left(1 - t^k \right)^\omega \mathrm{d}t \\ &= \frac{1}{\sqrt{k}} \int_0^1 (1 - z)^\omega z^{\frac{1}{k} - 1} \mathrm{d}z = \frac{1}{\sqrt{k}} \frac{\Gamma(1/k)\Gamma(\omega + 1)}{\Gamma(\omega + 1/k + 1)}, \end{aligned}$$

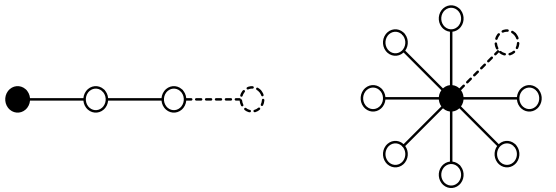
where the last equality follows from the definition of the Beta function, $B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} \mathrm{d}t$ for $\mathrm{Real}(x) > 0$, $\mathrm{Real}(y) > 0$, and its relation with the Gamma function, $B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$. ■

We now use Theorem 4.1 to investigate the asymptotic behavior of the expected observability radius on sequences of networks of increasing cardinality n. In order to emphasize the dependence on n, we shall write $\mathbb{E}[\delta(n)]$ from now on. As a first step, we can apply Wendel's inequalities [21] to find

$$\frac{1}{(\omega + 1)^{1/k}} \leq \frac{\Gamma(\omega + 1)}{\Gamma(\omega + 1 + 1/k)} \leq \frac{(\omega + 1 + 1/k)^{1 - 1/k}}{(\omega + 1)}.$$

If in a sequence of networks ω grows to infinity and k remains constant, then the ratio between the lower and the upper bounds goes to one, yielding the asymptotic equivalence

$$\mathbb{E}[\delta(n)] \leq \frac{\Gamma(1/k)\, \Gamma(\omega + 1)}{\sqrt{k}\, \Gamma(\omega + 1 + 1/k)} \sim \frac{\Gamma(1/k)}{\sqrt{k}} \frac{1}{(\omega + 1)^{1/k}}.$$



(a) Line network (b) Star network

Fig. 2. Line and star networks. Sensor nodes are marked in black.

This relation implies that a network becomes less robust to perturbations as the size of the network increases, with a rate determined by k. In the rest of this section we study two network topologies with different robustness properties. In particular, we show that line networks achieve the bound in Theorem 4.1, proving its tightness, whereas star networks have on average a smaller observability radius.

(Line network) Let $\mathcal{G}$ be a line network with n nodes and one sensor node as in Fig. 2. The adjacency and output matrices read as

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & \cdots & 0 & a_{n,n-1} & a_{nn} \end{bmatrix}, \quad (28)$$

$$C_{\mathcal{O}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

We obtain the following result.

Theorem 4.2: ***(Structured perturbation of line networks)*** Consider a line network with matrices as in (28), where the weights a_{ij} are independent random variables uniformly distributed in the interval $[0, 1]$. Let $\delta(n)$ be the minimal cost defined as in (27). Then,

$$\delta(n) = \min\{a_{12}, \ldots, a_{n-1,n}\}, \text{ and } \mathbb{E}[\delta(n)] = \frac{1}{n}.$$

Proof: It is known that line networks, when observed from one of their extremes, are strongly structurally observable, that is, they are observable for every nonzero choice of the edge weights [22]. Consequently, for the perturbed system to feature an unobservable eigenvalue, the perturbation Δ must be such that $\delta_{i,i+1} = -a_{i,i+1}$ for some $i \in \{2, \ldots, n-1\}$. Thus, a minimum norm perturbation is obtained by selecting the smallest entry $a_{i,i+1}$. Since the $a_{i,i+1}$ are independent and identically distributed, $\delta(n) = \min a_{i,i+1}$ is a random variable with survival function $\Pr(\delta(n) \geq x) = (1-x)^{n-1}$ for $0 \leq x \leq 1$, and $\Pr(\delta(n) \geq x) = 0$ otherwise. Thus,

$$\mathbb{E}[\delta(n)] = \int_0^1 \Pr(\delta(n) \geq x)\mathrm{d}x = \frac{1}{n}.$$

■

Theorem 4.2 characterizes the resilience of line networks to structured perturbations. We remark that, because line networks are strongly structurally observable, structured perturbations preventing observability necessarily disconnect the network by zeroing some network weights. Consistently with this remark, line networks achieve the upper bound in Theorem 4.1, being therefore maximally robust to structured perturbations. In fact, for $\mathcal{O} = \{1\}$ and a cut size $k = 1$ we have $\Omega_1(\mathcal{O}) = \{a_{12}, \ldots, a_{n-1,n}\}$ and $\omega = n-1$. Thus,

$$\mathbb{E}[\delta(n)] \leq \frac{\Gamma(1)\Gamma(n)}{\sqrt{1}\Gamma(n+1)} = \frac{(n-1)!}{n!} = \frac{1}{n},$$

which equals the behavior identified in Theorem 4.2. Further, Theorem 4.2 also identifies an unobservable eigenvalue yielding a perturbation with minimum norm. In fact, if $a_{i^*-1,i^*} = \min\{a_{12}, \ldots, a_{n-1,n}\}$, then all eigenvalues of the submatrix of A with rows/columns in the set $\{i^*, \ldots, n\}$ are unobservable, and thus minimizers in (27).

Both Theorems 4.1 and 4.2 are based on constructing perturbations by disconnecting the graph. This strategy, however, suffers from performance limitations and may not be optimal in general. The next example shows that different kinds of perturbations, when applicable, may yield a lower cost.

(Star network) Let $\mathcal{G}$ be a star network with n nodes and one sensor node as in Fig. 2. The adjacency and output matrices read as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & a_{n-1,n-1} & 0 \\ a_{n1} & 0 & 0 & 0 & a_{nn} \end{bmatrix}, \quad (29)$$

$$C_{\mathcal{O}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Differently from the case of line networks, star networks are not strongly structurally observable, so that different perturbations may result in unobservability of some modes.

Theorem 4.3: ***(Structured perturbation of star networks)*** Consider a star network with matrices as in (29), where the weights a_{ij} are independent random variables uniformly distributed in the interval $[0, 1]$. Let $\delta(n)$ be the minimal cost defined as in (27). Let

$$\gamma = \min_{i,j \in \{2,\ldots,n\}, i \neq j} \frac{|a_{ii} - a_{jj}|}{\sqrt{2}}.$$

Then,

$$\delta(n) = \min\{a_{12}, a_{13}, \ldots, a_{1n}, \gamma\}, \text{ and}$$

$$\frac{1}{\sqrt{2}\,n(n-1)} \leq \mathbb{E}[\delta(n)] \leq \frac{1}{\sqrt{2}\,n(n-2)}.$$

Proof: Partition the network matrix A in (29) as

$$A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{12} \in \mathbb{R}^{1,n-1}$, $A_{21} \in \mathbb{R}^{n-1,1}$, $A_{22} \in \mathbb{R}^{n-1,n-1}$. Accordingly, let $x = [x_1 \; x_2^\mathsf{T}]^\mathsf{T}$. The condition $C_{\mathcal{O}}x = 0$ implies $x_1 = 0$. Consequently, for the condition $(A + \Delta)x = \lambda x$ to be satisfied, we must have $(A_{12} + \Delta_{12})x_2 = 0$ and $(A_{22} + \Delta_{22})x_2 = \lambda x_2$. Notice that, because A_{22} is diagonal and $\Delta \in \mathcal{A}_{\mathcal{G}}$, the condition $(A_{22} + \Delta_{22})x_2 = \lambda x_2$ implies that $\lambda = a_{ii} + \delta_{ii}$ for all indices i such that $i \in \mathrm{Supp}(x_2)$, where $\mathrm{Supp}(x_2)$ denotes the set of nonzero entries of x_2. Because $\|x\| = 1$, $|\mathrm{Supp}(x_2)| > 0$. We have two cases:

Case $|Supp(x_2)| = 1$: Let $\mathrm{Supp}(x) = \{i\}$, with $i \in \{2, \ldots, n\}$. Then, the condition $(A_{12}+\Delta_{12})x_2 = 0$ implies $\delta_{1,i} = -a_{1,i}$, and the condition $(A_{22}+\Delta_{22})x_2 = \lambda x_2$ is satisfied with $\Delta_{22} = 0$, $\lambda = a_{ii}$, and $x = e_i$, where e_i is the i-th canonical vector of dimension n. Thus, if $|\mathrm{Supp}(x_2)| = 1$, then $\delta(n) = \min_{i \in \{2,\ldots,n\}} a_{1,i}$.

Case $|Supp(x_2)| > 1$: Let $S = \mathrm{Supp}(x_2)$. Then, $\delta_{ii} = \lambda - a_{ii}$. Notice that the condition $(A_{22}+\Delta_{22})x_2 = \lambda x_2$ is satisfied for every x_2 with support S and, particularly, for $x_2 \in \mathrm{Ker}(A_{12})$. Thus, we let $\Delta_{12} = 0$. Notice that

$$\delta(n) = \min_{\lambda, S} \sqrt{\sum_{i \in S} (\lambda - a_{ii})^2},$$

and that $\delta(n)$ is obtained when $S = \{i, j\}$, for some $i, j \in \{2, \ldots, n\}$, and $\lambda = (a_{ii} + a_{jj})/2$. Specifically, for the indexes $\{i, j\}$, we have $\|\Delta\|_\mathrm{F} = |a_{ii} - a_{jj}|/\sqrt{2}$. Thus, if $|\mathrm{Supp}(x_2)| > 1$, then $\delta(n) = \gamma$, which concludes the proof of the first statement.

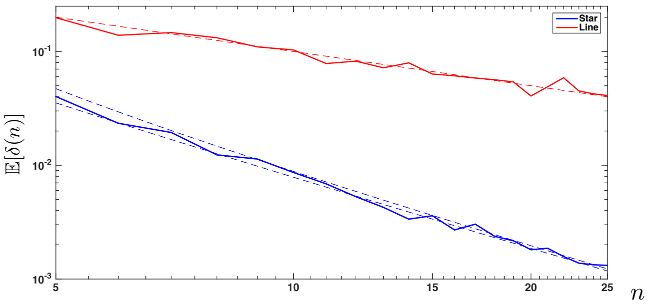


Fig. 3. Expected values $\mathbb{E}[\delta(n)]$ for the two network topologies in Fig. 2 as functions of the network cardinality n. Dotted lines represent upper and lower bounds in Theorems 4.2 and 4.3. Solid lines show the mean over 100 networks of the Frobenius norm of the perturbations obtained by Algorithm 1.

In order to estimate $\mathbb{E}[\delta(n)]$, notice that $\delta(n) = \min\{\alpha, \gamma\}$, where $\alpha = \min\{a_{12}, a_{13}, \dots, a_{1n}\}$, and that α and γ are independent random variables. Then, from [23, Chapter 6.4] we have

$$\begin{aligned}\Pr(\delta(n) \geq x) &= \Pr(\alpha \geq x)\Pr(\gamma \geq x)\\ &= (1-x)^{n-1}(1-(n-2)\sqrt{2}x)^{n-1},\end{aligned}$$

for $x \leq (\sqrt{2}(n-2))^{-1}$, and $\Pr(\delta(n) \geq x) = 0$ otherwise. Thus,

$$\mathbb{E}[\delta(n)] = \int_0^{\frac{1}{\sqrt{2}(n-2)}} (1-x)^{n-1}(1-(n-2)\sqrt{2}x)^{n-1}\mathrm{d}x.$$

Next, for the upper bound observe that

$$\begin{aligned}&\int_0^{\frac{1}{\sqrt{2}(n-2)}} (1-x)^{n-1}(1-(n-2)\sqrt{2}x)^{n-1}\mathrm{d}x\\ &\leq \int_0^{\frac{1}{\sqrt{2}(n-2)}} (1-(n-2)\sqrt{2}x)^{n-1}\mathrm{d}x = \frac{1}{\sqrt{2}n(n-2)},\end{aligned}$$

and for the lower bound observe that

$$\begin{aligned}&\int_0^{\frac{1}{\sqrt{2}(n-2)}} (1-x)^{n-1}(1-(n-2)\sqrt{2}x)^{n-1}\mathrm{d}x\\ &= \int_0^{\frac{1}{\sqrt{2}(n-2)}} (1-((n-2)\sqrt{2}+1)x + ((n-2)\sqrt{2})x^2)^{n-1}\mathrm{d}x\\ &\geq \int_0^{\frac{1}{\sqrt{2}(n-1)}} (1-(n-1)\sqrt{2}x)^{n-1}\mathrm{d}x = \frac{1}{\sqrt{2}n(n-1)}.\end{aligned}$$

■

Theorem 4.3 quantifies the resilience of star networks, and the unobservable eigenvalues requiring minimum norm perturbations; see the proof for a characterization of this eigenvalues.

The bounds in Theorem 4.3 are asymptotically tight and imply

$$\mathbb{E}[\delta(n)] \sim \frac{1}{\sqrt{2}\,n^2}, \quad \text{as } n \to \infty.$$

See Fig. 3 for a numerical validation of this result. This rate of decrease implies that star networks are *structurally* less robust to perturbations than line networks. Crucially, unobservability in star networks may be caused by two different phenomena: the deletion of an edge disconnecting a node from the sensor node (deletion of the smallest among the edges $\{a_{12}, a_{13}, \dots, a_{1n}\}$), and the creation of a dynamical symmetry with respect to the sensor node by perturbing two diagonal elements to make them equal in weight. It turns out that, on average, creating symmetries is "cheaper" than disconnecting the network. The role of network symmetries in preventing observability and controllability has been observed in several independent works; see for instance [16], [17]. Finally, the comparison of line and star networks shows that Algorithm 1 is a useful tool to systematically investigate the robustness of different topologies.

V. Conclusion

In this work we extend the notion of observability radius to network systems, thus providing a measure of the ability to maintain observability of the network modes against structured perturbations of the edge weights. We characterize network perturbations preventing observability, and describe a heuristic algorithm to compute perturbations with smallest Frobenius norm. Additionally, we study the observability radius of networks with random weights, derive a fundamental bound relating the observability radius to certain connectivity properties, and explicitly characterize the observability radius of line and star networks. Our results show that different network structures exhibit inherently different robustness properties, and thus provide guidelines for the design of robust complex networks.

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